On two constructions of extremal lattices

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Lattices: basic notions

A **lattice** $\Lambda \subset \mathbb{R}^n$ of rank $1 \leq k \leq n$ is a free $\mathbb{Z}$-module of rank k, which is the same as a discrete co-compact subgroup of $V := \operatorname{span}_{\mathbb{R}} \Lambda$. If $k = n$, i.e. $V = \mathbb{R}^n$, we say that Λ is a lattice of **full rank** in $\mathbb{R}^n$. Hence

$$\Lambda = \operatorname{span}_{\mathbb{Z}}\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_k\} = A\mathbb{Z}^k,$$

where $\boldsymbol{a}_1, \dots, \boldsymbol{a}_k \in \mathbb{R}^n$ are $\mathbb{R}$-linearly independent **basis** vectors for Λ and $A = (\boldsymbol{a}_1 \ \dots \ \boldsymbol{a}_k)$ is the corresponding $n \times k$ basis matrix.

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The **determinant** of Λ is

$$\det \Lambda := \sqrt{\det(A^t A)},$$

which is equal to the volume (quotient Lebesgue measure) of V/Λ.

Lattices: automorphisms

Let $\mathrm{GL}(\Lambda)$ be the subgroup of $\mathrm{GL}(V)$ that permutes Λ. The **automorphism group** of a lattice $\Lambda \subset \mathbb{R}^n$ is

$$\mathrm{Aut}(\Lambda) := \mathrm{GL}(\Lambda) \cap O(V),$$

where $\mathrm{GL}(\Lambda)$ is discrete and $O(V)$ is the compact group of orthogonal transformations of V onto itself $\Longrightarrow$ $\mathrm{Aut}(\Lambda)$ is finite.

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For all $n \neq 2,4,6,7,8,9,10$ the largest (with respect to order) automorphism group of a full rank lattice is

$$\mathrm{Aut}(\mathbb{Z}^n) = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n.$$

Lattices: minimal vectors

Minimal norm of a lattice Λ is

$$|\Lambda| = \min\left\{\|\boldsymbol{x}\| : \boldsymbol{x} \in \Lambda \setminus \{\boldsymbol{0}\}\right\},$$

where $\|\ \|$ is Euclidean norm. The set of **minimal vectors** of Λ is

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- It has been shown by Conway & Sloane (1995) and Martinet & Schürmann (2011) that there are lattices of rank ≥ 10 generated by minimal vectors which do not contain a **basis of minimal vectors**.

Lattices: eutaxy and perfection

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be the set of minimal vectors of the lattice Λ.

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This lattice is called **eutactic** if there exist positive real numbers $c_1, \dots, c_m$ such that

$$\|\boldsymbol{v}\|^2 = \sum_{i=1}^{m} c_i (\boldsymbol{v}, \boldsymbol{x}_i)^2$$

for every vector $\boldsymbol{v} \in \operatorname{span}_{\mathbb{R}} \Lambda$, where $(\cdot, \cdot)$ is the usual inner product. If $c_1 = \cdots = c_m$, we say that Λ is **strongly eutactic**.

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This lattice is called **perfect** if the set of symmetric matrices

$$\{\boldsymbol{x}_i \boldsymbol{x}_i^t : \boldsymbol{x}_i \in S(\Lambda)\}$$

spans the space of $k \times k$ symmetric matrices.

Packing density

The **packing density** of a lattice Λ of rank k is defined as

$$\delta(\Lambda) = \frac{\omega_k |\Lambda|^k}{2^k \det \Lambda},$$

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Remark 1

Non-perfect eutactic lattices are local minima of the packing density function.

Lattices from finite Abelian groups

Let

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Let

$$G = \{g_0 := 0, g_1, \ldots, g_{n-1}\}$$

be an Abelian group of order n, written additively.
We define a sublattice L_G of A_{n-1} by

$$L_G = \left\{ \boldsymbol{x} = (x_0, \ldots, x_{n-1}) \in A_{n-1} : \sum_{j=1}^{n-1} x_j g_j = 0 \right\}.$$

Results on Abelian group lattices

Theorem 2 (Böttcher, Eisenbarth, F., Garcia, Maharaj)

Let $n = |G|$ and let L_G be the lattice defined above. Then:

1. *For any G,* $\det L_G = n^{3/2}$.
2. $|L_G| = \begin{cases} \sqrt{8} & \text{if } G = \mathbb{Z}/2\mathbb{Z}, \\ \sqrt{6} & \text{if } G = \mathbb{Z}/3\mathbb{Z}, \\ 2 & \text{for any other } G. \end{cases}$
3. *For $G = \mathbb{Z}/4\mathbb{Z}$, the lattice L_G is not WR.*
4. *If $G \neq \mathbb{Z}/4\mathbb{Z}$, the lattice L_G has a basis of minimal vectors.*
5. *Let $\varepsilon = |\{g \in G : 2g = 0\}|$, then*
$$|S(L_G)| = \frac{n}{4\varepsilon}\left((n-\varepsilon)(n-\varepsilon-2) + n(n-2)(\varepsilon-1)\right).$$
6. *For any G,* $\mathrm{Aut}(L_G) \cap S_{n-1} \cong \mathrm{Aut}(G)$, *and* $\mathrm{Aut}(L_G) \cong (\mathbb{Z}/2\mathbb{Z}) \times (G \rtimes \mathrm{Aut}(G))$ *for $n = 3, 4, 6$ or $n \geq 12$.*

Eutaxy + perfection = extremality

Theorem 3 (Böttcher, Eisenbarth, F., Garcia, Maharaj)

The lattice L_G is strongly eutactic if and only if the Abelian group G has odd order or $G = (\mathbb{Z}/2\mathbb{Z})^k$ for some $k \geq 1$.

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Combining Bacher's result with ours and applying Voronoi's criterion, we obtain:

Corollary 5

If $|G| \geq 7$ is odd or $G = (\mathbb{Z}/2\mathbb{Z})^k$ for some $k \geq 3$, then L_G is extremal.

Equiangular tight frames (ETFs)

Another interesting construction of lattices comes from *frames*. A collection $\mathcal{F}$ of $n \geq k$ unit vectors $\boldsymbol{f}_1, \dots, \boldsymbol{f}_n \in \mathbb{R}^k$ is called an (real) (n, k)-**equiangular tight frame** (ETF) if it spans $\mathbb{R}^k$ and

1. $|(\boldsymbol{f}_i, \boldsymbol{f}_j)| = c(\mathcal{F})$ for all $1 \leq i \neq j \leq n$, for some constant $c(\mathcal{F}) \in [0, 1]$, called **coherence** of the frame $\mathcal{F}$,
2. $\sum_{i=1}^{n} (\boldsymbol{f}_i, \boldsymbol{x})^2 = \gamma \|\boldsymbol{x}\|^2$ for each $\boldsymbol{x} \in \mathbb{R}^k$, for some absolute constant $\gamma \in \mathbb{R}$.

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ETFs generalize the notion of an orthonormal basis, while redundancy of an overdetermined spanning set allows for better recovery of information in case of errors: we can think of "coordinates" with respect to such an overdetermined set as extra "frequencies" that can help recover information in case of erasures in transmission. ETFs are extensively used in coding theory and data compression, among many other areas – they are important tools of Applied Harmonic Analysis.

An example

Here is a $(3,2)$-ETF $\mathcal{F} := \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -\sqrt{3}/2 \\ -1/2 \end{pmatrix}, \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix} \right\}$:

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Notice that $\pm\mathcal{F} = S(\Lambda_h)$, the set of minimal vectors of the hexagonal lattice $\Lambda_h = \begin{pmatrix} 0 & \sqrt{3}/2 \\ 1 & 1/2 \end{pmatrix} \mathbb{Z}^2$.

Properties of ETFs

Let $\mathcal{F} = \{\boldsymbol{f}_1, \dots, \boldsymbol{f}_n\} \subset \mathbb{R}^k$ be an (n, k) real unit ETF. Here are some of its fundamental properties.

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3. (Neumann) If $n \neq 2k$, then $1/c(\mathcal{F})$ is an odd integer.
4. The constant γ is easily computed to be $\frac{k}{n}$.

Lattice construction

Let $\mathcal{F} = \{\boldsymbol{f}_1, \dots, \boldsymbol{f}_n\} \subset \mathbb{R}^k$ be a (n, k)-ETF, and define

$$\Lambda(\mathcal{F}) = \operatorname{span}_{\mathbb{Z}} \mathcal{F}.$$

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Question 1

When is $\Lambda(\mathcal{F})$ a lattice? If it is a lattice, what are its properties?

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Theorem 6 (Böttcher, F., Garcia, Maharaj, Needell)

1. *$\Lambda(\mathcal{F})$ is a lattice if and only if $c(\mathcal{F}) = \sqrt{\frac{n-k}{k(n-1)}}$ is rational.*
2. *If $\Lambda(\mathcal{F})$ is a lattice, it is of full rank.*
3. *If $\Lambda(\mathcal{F})$ is a lattice and*

$$S(\Lambda(\mathcal{F})) = \{\pm \boldsymbol{f}_1, \dots, \pm \boldsymbol{f}_n\},$$

then $\Lambda(\mathcal{F})$ is strongly eutactic.

Irrational example

Let $p = \frac{1+\sqrt{5}}{2}$ and let $\mathcal{F}$ be the set of columns of the matrix

$$\frac{1}{\sqrt{1+p^2}} \begin{pmatrix} 0 & 0 & 1 & -1 & p & p \\ 1 & -1 & p & p & 0 & 0 \\ p & p & 0 & 0 & 1 & -1 \end{pmatrix}.$$

This is a $(6,3)$ real unit ETF with *irrational* coherence $1/\sqrt{5} \approx 0.4472$. By Dirichlet's theorem in Diophantine Approximations, there exist infinitely many relatively prime integers a, b such that

$$\left| \frac{a}{b} + p \right| \leq \frac{1}{b^2}.$$

Taking a linear combination of the vectors of $\mathcal{F}$ with coefficients

$$a+b,\ a-b,\ b,\ b,\ a,\ -a,$$

we obtain a constant multiple of a vector $(0, a+bp, a+bp)^\top$. Coordinates of this vector are $\leq 1/b$ in absolute value, so it $\to \mathbf{0}$ as $b \to \infty$, i.e. $\Lambda(\mathcal{F})$ is not discrete.

Summary of our results

Table: There exists an (n, k) real unit ETF $\mathcal{F}$ with:

(n, k)	$c(\mathcal{F})$	**Eutactic?**	**Perfect?**	$\delta(\Lambda(\mathcal{F}))$	$\delta_{\max}$
$(k+1, k)$	$1/k$	strongly	only $(3, 2)$	–	–
$\mathbf{(6, 3)}$	$\mathbf{1/\sqrt{5}}$	**n/a**	**n/a**	**n/a**	**n/a**
$(10, 5)$	$1/3$	strongly	no	0.3701...	0.4652...
$(16, 6)$	$1/3$	strongly	no	0.2725...	0.3729...
$\mathbf{(14, 7)}$	$\mathbf{1/\sqrt{13}}$	**n/a**	**n/a**	**n/a**	**n/a**
$(28, 7)$	$1/3$	strongly	yes	0.2157...	0.2953...
$\mathbf{(18, 9)}$	$\mathbf{1/\sqrt{17}}$	**n/a**	**n/a**	**n/a**	**n/a**
$(26, 13)$	$1/5$	strongly	no	0.0024...	0.0320...
$(276, 23)$	$1/5$	yes	yes	?	0.0019...
$(50, 25)$	$1/7$	?	no	–	–

$\delta_{\max}$ = max packing density known in $\mathbb{R}^k$.

Further results on ETF lattices

Theorem 7 (Böttcher, F., Garcia, Maharaj, Needell)

1. *Lattices* $\Lambda(\mathcal{F})$ *from the* $(k+1, k)$, $(16, 6)$, $(28, 7)$ *entries of Table 1 all have bases of minimal vectors.*
2. *There are infinitely many* k *for which there exist* $(2k, k)$*-ETFs* $\mathcal{F}$ *such that* $\Lambda(\mathcal{F})$ *is a full-rank lattice, e.g.* $(10, 5)$, $(26, 13)$.
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Remark 2

1. There are often multiple ETFs with the same parameters (n,k): we exhibit two lattices from $(10,5)$-ETFs, three lattices from $(26,13)$-ETFs, and ten lattices from $(50,25)$-ETFs.

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2. Perfection of the lattice from $(28, 7)$-ETF (constructed differently) was established in 2015 by R. Bacher.

More remarks

Minimal vectors of ETF lattices are often precisely the $\pm$ frame vectors (this is the case with all our examples). The corresponding symmetric matrices are known to be linearly independent, and so there are

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In case of equality, we likely *always* get a perfect strongly eutactic (hence extremal) lattice with the minimal required number of minimal vectors for the perfection condition. This being said, the only known cases of equality in the Gerzon bound are $(3, 2)$, $(6, 3)$, $(28, 7)$ and $(276, 23)$; the $(3, 2)$ case is precisely the famous planar hexagonal lattice.

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`http://math.cmc.edu/lenny/research.html`