On the geometry of period lattices for elliptic curves

Lenny Fukshansky
Claremont McKenna College

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To the memory of Florian Luca

1969 - 2026

Similarity classes of planar lattices

Every $A \in \mathrm{GL}_2(\mathbb{R})$ is a basis matrix for some planar lattice

$$\Omega := A\mathbb{Z}^2 = AB\mathbb{Z}^2,$$

for any $B \in \mathrm{GL}_2(\mathbb{Z})$. Hence the space of planar lattices $\mathcal{L}_2$ can be identified with $\mathrm{GL}_2(\mathbb{R})/\mathrm{GL}_2(\mathbb{Z})$. Two lattices Ω and Γ are said to be **similar**, denoted $\Omega \sim \Gamma$, if $\Omega = \alpha U \Gamma$ for some positive real constant α and orthogonal matrix U.

Every lattice $\Omega \in \mathcal{L}_2$ is similar to a unique lattice of the form

$$\Gamma_\tau := \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \mathbb{Z}^2 \text{ for some } \tau := a + bi \text{ in}$$

$$\mathcal{F} := \{\tau = a + bi \in \mathbb{C} : 0 \leq a \leq 1/2, b \geq 0, |\tau| \geq 1\}.$$

We refer to $\mathcal{F}$ as the **set of similarity classes** of lattices in $\mathcal{L}_2$.

Elliptic curves and isogenies

Given lattices $\Lambda, \Lambda' \subset \mathbb{C}$ a nonzero morphism $E \to E'$ between the corresponding elliptic curves $E = \mathbb{C}/\Lambda$ and $E' = \mathbb{C}/\Lambda'$ which takes 0 to 0 is called an **isogeny**. An isogeny is always surjective and has a finite kernel. For instance, if $\beta \in \mathbb{C}^*$ is such that $\beta\Lambda \subseteq \Lambda'$, then multiplication by β function $z \mapsto \beta z$ maps $\mathbb{C} \to \mathbb{C}$ modulo the lattices, hence maps $E \to E'$. In fact, every isogeny is of this form. The **degree** of this isogeny is the degree of the morphism, which is equal to the index of the sublattice $\beta\Lambda$ in Λ', i.e.

$$\deg(\beta) = [\Lambda' : \beta\Lambda].$$

This is precisely the size of its kernel. If an isogeny $E \to E'$ exists, then there also exists the dual isogeny $E' \to E$ of the same degree such that their composition is the multiplication-by-degree map, and hence the curves are called **isogenous**: this is an equivalence relation. There may exist multiple isogenies between two elliptic curves, but since degree of an isogeny is a positive integer, we can ask for an isogeny of minimal degree.

Isomorphism classes of elliptic curves

An **isomorphism** of elliptic curves is an injective isogeny, i.e. of degree one. Each elliptic curve is isomorphic to an elliptic curve E_τ with period lattice

$$\Gamma_\tau = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \mathbb{Z}^2 \text{ for some } \tau := a + bi \text{ in}$$

$$\mathcal{D} := \{\tau = a + bi \in \mathbb{C} : -1/2 < a \leq 1/2, b \geq 0, |\tau| \geq 1\}.$$

Further, $\mathcal{D}' := \mathcal{D} \setminus \{e^{i\theta} : \pi/2 < \theta < 2\pi/3\}$ is precisely the **set of isomorphism classes** of elliptic curves.

This set $\mathcal{D}$ can also be viewed as a fundamental domain for the action of the group $\mathrm{SL}_2(\mathbb{Z})$ on the set of lattices Γ_τ by right matrix multiplication by g^{-1} for each $g \in SL_2(\mathbb{Z})$:

$$\Gamma_\tau = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \mathbb{Z}^2 \mapsto g \cdot \Gamma_\tau := \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} g^{-1} \mathbb{Z}^2.$$

Arithmetic, well-rounded, semi-stable lattices

A lattice $\Gamma = A\mathbb{Z}^2$ is called **arithmetic** if the matrix $A^\top A$ is a scalar multiple of an integral matrix: this property is independent of the choice of the basis matrix A.

Successive minima of Γ are real numbers $0 < \lambda_1(\Gamma) \leq \lambda_2(\Gamma)$:

$$\lambda_j(\Gamma) := \min\left\{r \in \mathbb{R}_{>0} : \dim_{\mathbb{R}} \operatorname{span}_{\mathbb{R}}\left(\mathbb{B}(r) \cap L\right) \geq j\right\},$$

where $\mathbb{B}(r)$ is the disk of radius r centered at the origin in $\mathbb{R}^2$. Γ is called **well-rounded (WR)** if $\lambda_1(\Gamma) = \lambda_2(\Gamma)$.

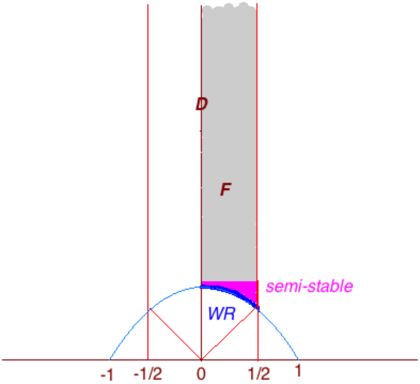
Γ is called **semi-stable** if

$$\lambda_1(L)^2 \geq \det(\Gamma) := |\det(A)|.$$

WR and semi-stable lattices are important in discrete optimization, reduction theory, and related areas.

Geometrically speaking...

These properties of lattices are constant on similarity classes, hence we speak of arithmetic, WR, semi-stable similarity classes in $\mathcal{L}_2$, and therefore in $\mathcal{F}$.



Algebraically speaking...

Γ_τ is arithmetic iff $\tau \in \mathcal{F}$ is of the form

$$\tau = \tau(a, b, c, d) := \frac{a}{b} + i\sqrt{\frac{c}{d}}$$

for some integers a, b, c, d such that

$$\gcd(a, b) = \gcd(c, d) = 1,\ 0 \leq a \leq b/2,\ c/d \geq 1 - a^2/b^2.$$

This condition is equivalent to the elliptic curve E_τ with period lattice Γ_τ having **complex multiplication** (CM) by the imaginary quadratic field $\mathbb{Q}(\tau)$, i.e. the endomorphism ring of E_τ is an order in $\mathbb{Q}(\tau)$ properly containing $\mathbb{Z}$.

The j-invariant

The **Klein j-function** is a bijective holomorphic map $j : \mathcal{D}' \to \mathbb{C}$. If E is an elliptic curve, then it is isomorphic to an elliptic curve E_τ for precisely one $\tau \in \mathcal{D}'$, and hence the value $j(\tau)$ is an invariant of this elliptic curve, called its **j-invariant**.

Here are some properties of the j-invariant in terms of the corresponding lattices:

- For $\tau \in \mathcal{D}$, $j(\tau) \in \mathbb{R}$ iff τ belongs to the boundary of $\mathcal{F}$, and Γ_τ is WR iff $j(\tau) \in [0,1]$.
- Suppose $\tau \in \mathcal{F}$ is algebraic. Then

$$\Gamma_\tau \text{ is arithmetic } \iff \deg_{\mathbb{Q}}(\tau) = 2 \iff j(\tau) \in \overline{\mathbb{Q}}.$$

  In this case, the degree of the algebraic number $j(\tau)$ is the class number of the quadratic imaginary number field $\mathbb{Q}(\tau)$.

Maximum height

We define the **maximum height** of an arithmetic similarity class $\Gamma_{\tau(a,b,c,d)}$ to be

$$\mathfrak{m}(\Gamma_{\tau(a,b,c,d)}) = \mathfrak{m}(\tau(a,b,c,d)) := \max\{|a|,|b|,|c|,|d|\}.$$

Then, the number of arithmetic similarity classes with $\mathfrak{m}(\Gamma_{\tau(a,b,c,d)}) \leq T$ is finite for every real number T.

Let $T \in \mathbb{Z}_{>0}$, and define

$$\begin{aligned} N_1(T) &= \left|\{\Lambda_\tau : \Lambda_\tau \text{ is arithmetic and } \mathfrak{m}(\Lambda_\tau) \leq T\}\right|, \\ N_2(T) &= \left|\{\Lambda_\tau : \Lambda_\tau \text{ is arithmetic semi-stable and } \mathfrak{m}(\Lambda_\tau) \leq T\}\right|, \\ N_3(T) &= \left|\{\Lambda_\tau : \Lambda_\tau \text{ is arithmetic WR and } \mathfrak{m}(\Lambda_\tau) \leq T\}\right|. \end{aligned}$$

Counting estimate

Theorem 1 (F., Guerzhoy, Luca (2015))

With notation as above, $N_1(T) > N_2(T) > N_3(T)$, and as $T \to \infty$,

$$N_1(T) = \frac{39T^4}{8\pi^4} + O(T^3 \log T),$$

and

$$N_2(T) = \frac{3T^4}{8\pi^4} + O(T^3 \log T),$$

while

$$N_3(T) = \frac{3T^2}{2\pi^2} + O(T \log T).$$

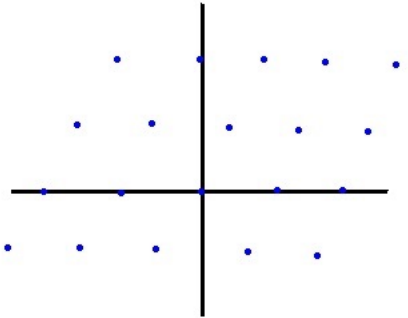
Deep holes

Let $L \subset \mathbb{R}^2$ be a lattice with successive minima $\lambda_1 \leq \lambda_2$ and the corresponding minimal basis vectors $\boldsymbol{x}_1, \boldsymbol{x}_2$. It is well known that, choosing $\pm\boldsymbol{x}_1, \pm\boldsymbol{x}_2$ if necessary, we can ensure that the angle θ between these vectors is in the interval $[\pi/3, \pi/2]$: this angle is an invariant of the lattice, we call it **angle** of L.

A **deep hole** of L is a point in $\mathbb{R}^2$ which is farthest away from the lattice. The distance from the origin to the nearest deep hole is the *covering radius* μ of L. There is a unique deep hole $\boldsymbol{z}$ of L contained in the triangle T with vertices $\mathbf{0}$ and the endpoints of $\boldsymbol{x}_1, \boldsymbol{x}_2$: we call it the *fundamental deep hole* of L. Define the **deep hole lattice** of L to be

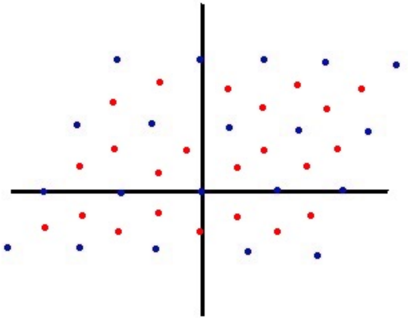
$$H(L) := \operatorname{span}_{\mathbb{Z}}\{\boldsymbol{x}_1, \boldsymbol{z}\}.$$

Deep holes



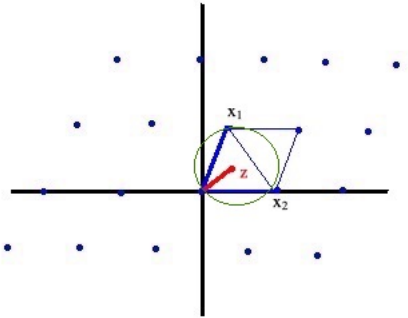
Lattice points in blue

Deep holes



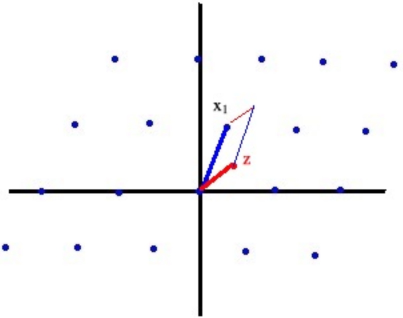
Deep holes in red

Deep holes



Fundamental deep hole

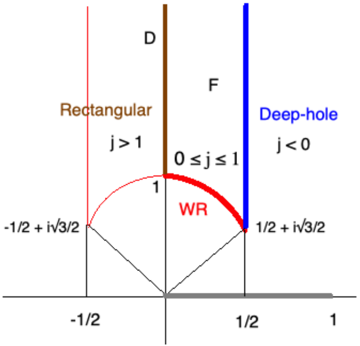
Deep holes



Deep hole lattice: $H(L) = \operatorname{span}_{\mathbb{Z}}\{\boldsymbol{x}_1, \boldsymbol{z}\}$.

Back to the fundamental strip

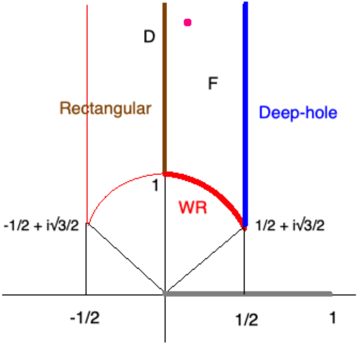
A lattice is **deep-hole** if it is similar to $H(L)$ for some L. A lattice is **rectangular** if it contains an orthogonal basis.



Boundary $\partial\mathcal{F} =$ rectangular $\cup$ WR $\cup$ deep-hole

Deep-hole sequence

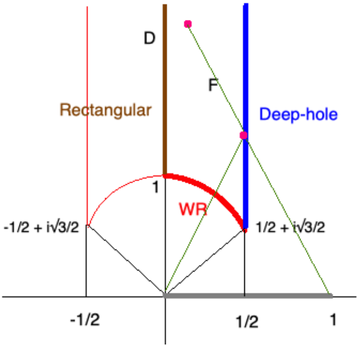
A lattice is **deep-hole** if it is similar to $H(L)$ for some L. A lattice is **rectangular** if it contains an orthogonal basis.



Arbitrary similarity class

Deep-hole sequence

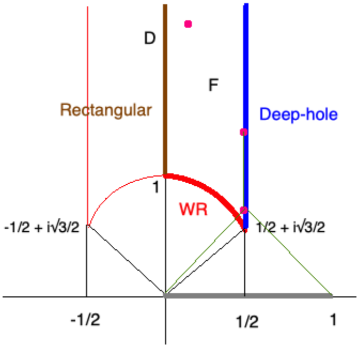
A lattice is **deep-hole** if it is similar to $H(L)$ for some L. A lattice is **rectangular** if it contains an orthogonal basis.



Its deep hole lattice

Deep-hole sequence

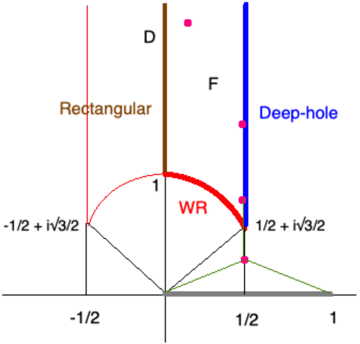
A lattice is **deep-hole** if it is similar to $H(L)$ for some L. A lattice is **rectangular** if it contains an orthogonal basis.



Next deep hole step

Deep-hole sequence

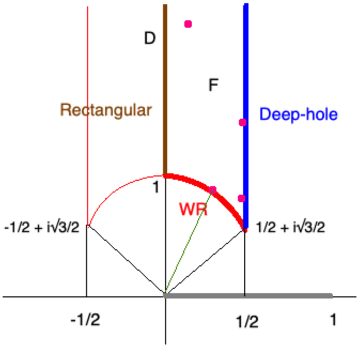
A lattice is **deep-hole** if it is similar to $H(L)$ for some L. A lattice is **rectangular** if it contains an orthogonal basis.



Final deep hole step

Deep-hole sequence

A lattice is **deep-hole** if it is similar to $H(L)$ for some L. A lattice is **rectangular** if it contains an orthogonal basis.



It is similar to a WR lattice, which is similar to its deep hole lattice

Properties of this *deep hole sequence* were studied by **F., Guerzhoy, and Nielsen (2024)**.

Commensurable lattices

Two planar lattices Ω and Γ are **commensurable**, denoted $\Omega \simeq \Gamma$, if one contains a sublattice similar to the other one. This is also an equivalence relation.

Lattices are similar $\Leftrightarrow$ elliptic curves are isomorphic

Lattices are commensurable $\Leftrightarrow$ elliptic curves are isogenous

Commensurability on the boundary: Each rectangular lattice is commensurable with *its* deep hole lattice and with a WR lattice.

Proposition 2 (F., Knight (2026+))

Two planar lattices Ω and Γ are commensurable if and only if they have the same angle sets, i.e.,

$$\{\angle(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{x}, \boldsymbol{y} \in \Omega\} = \{\angle(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{x}, \boldsymbol{y} \in \Lambda\},$$

where $\angle(\boldsymbol{x}, \boldsymbol{y})$ stands for the angle between two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.

Isogenies of elliptic curves

Theorem 3 (F., Guerzhoy, Kühnlein (2020))

Let $\tau = a + bi \in \mathcal{D}$ and let E_τ be the corresponding elliptic curve with the period lattice Γ_τ. The following are equivalent:

1. *Either $a \in \mathbb{Q}$ or there exists some $t \in \mathbb{R}$ such that $\frac{v'}{v} := a - bt$, $\frac{w'}{w} := a + b/t \in \mathbb{Q}$,*
2. *Γ_τ is commensurable to a rectangular lattice, to a WR lattice, and to a deep-hole lattice,*
3. *E_τ is isogenous to E' with real j-invariant.*

If (1) - (3) hold with $a = p/q \in \mathbb{Q}$, then exists an isogeny $E' \to E_\tau$ with degree $\delta(E'/E_\tau) = q$. If (1) - (3) hold with $a \notin \mathbb{Q}$ and $t \in \mathbb{R}$ satisfies (1), then exists an isogeny $E' \to E_\tau$ with

$$\delta(E'/E_\tau) = \frac{|b||vw|(t^2+1)}{|t|}.$$

Commensurability index

Suppose two lattices $L, M \subset \mathbb{R}^2$ are commensurable. Define their mutual **commensurability index** to be

$$\begin{aligned} \mathrm{ind}(L, M) &= \min\left\{[L : L'] : L' \subset L, L' \sim M\right\} \\ &= \min\left\{[M : M'] : M' \subset M, M' \sim L\right\} = \mathrm{ind}(M, L). \end{aligned}$$

The fact that $\mathrm{ind}(L, M) = \mathrm{ind}(M, L)$ is a 2-dimensional phenomena: analogous commensurability indices $\mathrm{ind}(L, M)$ and $\mathrm{ind}(M, L)$ for lattices in higher dimensions are not necessarily equal.

Let E_L, E_M be isogenous elliptic curves with commensurable period lattices L, M. Then the minimal degree of an isogeny

$$\delta(E_L/E_M) = \delta(E_M/E_L) = \mathrm{ind}(L, M) = \mathrm{ind}(M, L).$$

Commensurability density

Question 1

How are commensurability classes distributed in the space of similarity classes?

Theorem 4 (F., Knight (2026+))

Let $\tau_1 = a_1 + b_1 i \in \mathcal{F}$*. Then the set of* $\tau_0 \in \partial\mathcal{F}$ *such that* $\Gamma_{\tau_1} \simeq \Gamma_{\tau_0}$ *is dense in* $\partial\mathcal{F}$*, and the set of* $\tau_0 \in \mathcal{F}$ *such that* $\Gamma_{\tau_1} \simeq \Gamma_{\tau_0}$ *is dense in* $\mathcal{F}$*. Further, for any* $\tau_2 = a_2 + b_2 i \in \mathcal{F}$ *and* $\varepsilon > 0$*, there exists* $\tau_0 \in \mathcal{F}$ *such that* $|\tau_2 - \tau_0| < \varepsilon$*,* $\Gamma_{\tau_1} \simeq \Gamma_{\tau_0}$ *and*

$$\operatorname{ind}(\Gamma_{\tau_1}, \Gamma_{\tau_0}) \leq \frac{2(\varepsilon^2 + 2b_1 b_2)}{\varepsilon^4}.$$

So, every commensurability class of lattices (isogeny class of elliptic curves) is dense among similarity classes of lattices (isomorphism classes of elliptic curves).

References

L. Fukshansky, P. Guerzhoy, F. Luca. *On arithmetic lattices in the plane*, Proceedings of the American Mathematical Society, vol. 145 no. 4 (2017), pg. 1453–1465

L. Fukshansky, P. Guerzhoy, S. Kühnlein. *On sparse geometry of numbers*, Research in the Mathematical Sciences, vol. 8 no. 1 (2021), Article #2

L. Fukshansky, P. Guerzhoy, T. Nielsen. *Deep hole lattices and isogenies of elliptic curves*, Research in Number Theory, vol. 10 no. 2 (2024), Article#33, 12 pp.

L. Fukshansky, E. Knight. *Commensurability of lattices in the plane*, in progress, (2026+)

http://math.cmc.edu/lenny/research.html

Thank you!