Geometric constructions for sparse integer signal recovery

Lenny Fukshansky
Claremont McKenna College

UC Irvine – Combinatorics Seminar
February 16, 2022

Sensing matrices

An $n \times d$ real matrix A is said to be a **sensing matrix for ℓ-sparse signals**, $1 \leq \ell \leq n$, if for every nonzero vector $\boldsymbol{x} \in \mathbb{R}^d$ with no more than ℓ nonzero coordinates, $A\boldsymbol{x} \neq \mathbf{0}$.

Sensing matrices

An $n \times d$ real matrix A is said to be a **sensing matrix for ℓ-sparse signals**, $1 \leq \ell \leq n$, if for every nonzero vector $\boldsymbol{x} \in \mathbb{R}^d$ with no more than ℓ nonzero coordinates, $A\boldsymbol{x} \neq \mathbf{0}$.

This is equivalent to saying that no ℓ columns of A are linearly dependent: such matrices $A = (a_{ij})$ are extensively used in the area of compressive sensing for the purposes of robust signal recovery from few measurements.

Sensing matrices

An $n \times d$ real matrix A is said to be a **sensing matrix for ℓ-sparse signals**, $1 \leq \ell \leq n$, if for every nonzero vector $\boldsymbol{x} \in \mathbb{R}^d$ with no more than ℓ nonzero coordinates, $A\boldsymbol{x} \neq \boldsymbol{0}$.

This is equivalent to saying that no ℓ columns of A are linearly dependent: such matrices $A = (a_{ij})$ are extensively used in the area of compressive sensing for the purposes of robust signal recovery from few measurements.

We will specifically be interested in integer sensing matrices. The goal is to have d as large as possible with respect to n while

$$|A| := \max |a_{ij}|$$

is small.

Sensing matrices

An $n \times d$ real matrix A is said to be a **sensing matrix for ℓ-sparse signals**, $1 \leq \ell \leq n$, if for every nonzero vector $\boldsymbol{x} \in \mathbb{R}^d$ with no more than ℓ nonzero coordinates, $A\boldsymbol{x} \neq \mathbf{0}$.

This is equivalent to saying that no ℓ columns of A are linearly dependent: such matrices $A = (a_{ij})$ are extensively used in the area of compressive sensing for the purposes of robust signal recovery from few measurements.

We will specifically be interested in integer sensing matrices. The goal is to have d as large as possible with respect to n while

$$|A| := \max |a_{ij}|$$

is small. If we have such a matrix A and two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ with no more than $\ell/2$ nonzero coordinates each, then it is easy to see that $A\boldsymbol{x} = A\boldsymbol{y}$ if and only if $\boldsymbol{x} = \boldsymbol{y}$.

Sensing matrices

The advantage of using integer matrices and integer signals is that in this situation

$$A\boldsymbol{x} \neq \mathbf{0} \implies \|A\boldsymbol{x}\| \geq 1,$$

which allows for robust error correction.

Sensing matrices

The advantage of using integer matrices and integer signals is that in this situation

$$A\mathbf{x} \neq \mathbf{0} \implies \|A\mathbf{x}\| \geq 1,$$

which allows for robust error correction. Hence we have the following optimization problem:

Problem 1

Construct an $n \times d$ integer sensing matrix A for ℓ-sparse signals such that:

1. $|A|$ *is small,*
2. d *is large compared to* n*,*
3. $\ell \leq n$ *is as large as possible.*

Existence results – I

Theorem 1 (F., Needell, Sudakov – 2017)

For all sufficiently large n there exist $n \times d$ integer sensing matrices A for n-sparse signals with

$$|A| = 1 \text{ and } d = 1.2938\, n.$$

More generally, there exist such matrices with

$$|A| = k \text{ and } d = \Omega\left(\sqrt{k}\, n\right).$$

Existence results – I

Theorem 1 (F., Needell, Sudakov – 2017)

For all sufficiently large n there exist $n \times d$ integer sensing matrices A for n-sparse signals with

$$|A| = 1 \text{ and } d = 1.2938\, n.$$

More generally, there exist such matrices with

$$|A| = k \text{ and } d = \Omega\left(\sqrt{k}\, n\right).$$

We outline our proof of this theorem, which is probabilistic in nature.

Proof of Theorem 1

First we need to prove the existence of an $n \times d$ matrix A with

$$|A| = 1 \text{ and } d = 1.2938\, n,$$

so that every $n \times n$ submatrix is nonsingular.

Proof of Theorem 1

First we need to prove the existence of an $n \times d$ matrix A with

$$|A| = 1 \text{ and } d = 1.2938\, n,$$

so that every $n \times n$ submatrix is nonsingular.

We use a powerful result of Bourgain, Vu and Wood (2010):

Let M_n be an $n \times n$ random matrix whose entries are 0 with probability $1/2$ and ± 1 with probability $1/4$ each. Then the probability that matrix M_n is singular is at most $(1/2 - o(1))^n$.

Proof of Theorem 1

First we need to prove the existence of an $n \times d$ matrix A with

$$|A| = 1 \text{ and } d = 1.2938\, n,$$

so that every $n \times n$ submatrix is nonsingular.

We use a powerful result of Bourgain, Vu and Wood (2010):

Let M_n be an $n \times n$ random matrix whose entries are 0 with probability $1/2$ and ± 1 with probability $1/4$ each. Then the probability that matrix M_n is singular is at most $(1/2 - o(1))^n$.

Form an $n \times d$ random matrix A by taking its entries to be 0 with probability $1/2$ and ± 1 with probability $1/4$ each. Then $|A| = 1$ and any n columns of A form a matrix distributed according to M_n.

Proof of Theorem 1

Therefore the probability that any $n \times n$ submatrix of A is singular is at most

$$(1/2 - o(1))^n.$$

Since the number of such submatrices is $\binom{d}{n}$ we have (by union bound) that the probability that A contains an $n \times n$ singular submatrix is at most

$$\binom{d}{n}(1/2 - o(1))^n.$$

Proof of Theorem 1

Therefore the probability that any $n \times n$ submatrix of A is singular is at most

$$(1/2 - o(1))^n.$$

Since the number of such submatrices is $\binom{d}{n}$ we have (by union bound) that the probability that A contains an $n \times n$ singular submatrix is at most

$$\binom{d}{n}(1/2 - o(1))^n.$$

Bounding $\binom{d}{n}$ we see that this probability is < 1 for $d \leq 1.2938\,n$, and hence an $n \times d$ matrix A with $|A| = 1$ and all $n \times n$ submatrices nonsingular exists.

Proof of Theorem 1

Next, we want to prove existence of an $n \times d$ matrix A with

$$|A| = k \text{ and } d = \Omega(\sqrt{k}\, n),$$

so that every $n \times n$ submatrix is nonsingular.

Proof of Theorem 1

Next, we want to prove existence of an $n \times d$ matrix A with

$$|A| = k \text{ and } d = \Omega(\sqrt{k}\, n),$$

so that every $n \times n$ submatrix is nonsingular.

We use another result of Bourgain, Vu and Wood (2010) from the same paper:

If N_n is an $n \times n$ random matrix whose entries come from the set $\{-k, \cdots, k\}$ with equal probability, then the probability that N_n is singular is at most $(1/\sqrt{2k} - o(1))^n$.

Proof of Theorem 1

Next, we want to prove existence of an $n \times d$ matrix A with

$$|A| = k \text{ and } d = \Omega(\sqrt{k}\, n),$$

so that every $n \times n$ submatrix is nonsingular.

We use another result of Bourgain, Vu and Wood (2010) from the same paper:

If N_n is an $n \times n$ random matrix whose entries come from the set $\{-k, \cdots, k\}$ with equal probability, then the probability that N_n is singular is at most $(1/\sqrt{2k} - o(1))^n$.

Consider an $n \times d$ random matrix A whose entries come from the set $\{-k, \cdots, k\}$ with equal probability. Then $|A| \leq k$ and the probability that any $n \times n$ submatrix of A is singular is at most

$$(1/\sqrt{2k} - o(1))^n.$$

Proof of Theorem 1

Since the number of such submatrices is $\binom{d}{n}$ we have that the probability that A contains an $n \times n$ singular submatrix is at most

$$\binom{d}{n}\left(\frac{1}{\sqrt{2k}} - o(1)\right)^n .$$

Proof of Theorem 1

Since the number of such submatrices is $\binom{d}{n}$ we have that the probability that A contains an $n \times n$ singular submatrix is at most

$$\binom{d}{n}\left(\frac{1}{\sqrt{2k}} - o(1)\right)^n.$$

Again, estimating the binomial coefficient and choosing d to be a sufficiently small multiple of $\sqrt{k}n$, this probability can be made smaller than 1.

Proof of Theorem 1

Since the number of such submatrices is $\binom{d}{n}$ we have that the probability that A contains an $n \times n$ singular submatrix is at most

$$\binom{d}{n}\left(\frac{1}{\sqrt{2k}} - o(1)\right)^n .$$

Again, estimating the binomial coefficient and choosing d to be a sufficiently small multiple of $\sqrt{k}n$, this probability can be made smaller than 1.

Thus with positive probability A does not have singular $n \times n$ submatrices.

How much bigger than n can d be?

Theorem 2 (F., Needell, Sudakov – 2017)

For any integers $n \geq 3$, $k \geq 1$ and $n \times d$ integer sensing matrix A for n-sparse signals with $|A| = k$, we must have

$$d \leq (2k^2 + 2)(n - 1) + 1.$$

How much bigger than n can d be?

Theorem 2 (F., Needell, Sudakov – 2017)

For any integers $n \geq 3$, $k \geq 1$ and $n \times d$ integer sensing matrix A for n-sparse signals with $|A| = k$, we must have

$$d \leq (2k^2 + 2)(n - 1) + 1.$$

Theorem 3 (Konyagin, Sudakov – 2018)

For sufficiently large n,

$$d = O(k\sqrt{\log k}\ n).$$

How much bigger than n can d be?

Theorem 2 (F., Needell, Sudakov – 2017)

For any integers $n \geq 3$, $k \geq 1$ and $n \times d$ integer sensing matrix A for n-sparse signals with $|A| = k$, we must have

$$d \leq (2k^2 + 2)(n - 1) + 1.$$

Theorem 3 (Konyagin, Sudakov – 2018)

For sufficiently large n,

$$d = O(k\sqrt{\log k}\; n).$$

Konyagin and Sudakov give a deterministic construction of such matrices with

$$d \geq \frac{k^{\frac{n}{n-1}}}{2} > n.$$

Question and example

Question 1

What is the optimal upper bound on d in terms of k and n? In particular, is it true that $d = O(kn)$?

Question and example

Question 1

What is the optimal upper bound on d in terms of k and n? In particular, is it true that $d = O(kn)$?

Here is a low-dimensional example.

Example 1

Let $n = 3$, $d = 6$, $k = 1$, and define a 3×6 matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & 0 & 1 & -1 & 0 & -1 \end{pmatrix}.$$

This matrix has $|A| = 1$ and any three of its columns are linearly independent.

Existence results – II

What if we allow the sparsity level ℓ to be smaller than n?

Existence results – II

What if we allow the sparsity level ℓ to be smaller than n?

Theorem 4 (F., Hsu – 2019)

For all sufficiently large n, there exist $n \times d$ integer sensing matrices A for ℓ-sparse signals, $1 \leq \ell \leq n-1$, such that $|A| = 2$ and

$$d \geq \left(\frac{n+2}{2}\right)^{1+\frac{2}{3\ell-2}}.$$

Existence results – II

What if we allow the sparsity level ℓ to be smaller than n?

Theorem 4 (F., Hsu – 2019)

For all sufficiently large n, there exist $n \times d$ integer sensing matrices A for ℓ-sparse signals, $1 \leq \ell \leq n-1$, such that $|A| = 2$ and

$$d \geq \left(\frac{n+2}{2}\right)^{1+\frac{2}{3\ell-2}} .$$

An important implication of Theorem 4 is that when $\ell = o(\log n)$, then $d/n \to \infty$ as $n \to \infty$, meaning that d is super-linear in n.

Existence results – II

What if we allow the sparsity level ℓ to be smaller than n?

Theorem 4 (F., Hsu – 2019)

For all sufficiently large n, there exist $n \times d$ integer sensing matrices A for ℓ-sparse signals, $1 \leq \ell \leq n-1$, such that $|A| = 2$ and

$$d \geq \left(\frac{n+2}{2}\right)^{1+\frac{2}{3\ell-2}}.$$

An important implication of Theorem 4 is that when $\ell = o(\log n)$, then $d/n \to \infty$ as $n \to \infty$, meaning that d is super-linear in n.

Our construction of sensing matrices is based on a particular geometric covering problem, which is what we discuss next.

Covering by parallel hyperplanes

Suppose $S \subset \mathbb{R}^n$ is a set of k points.

It is not difficult to notice that S can always be covered by no more than $\max\{1, k-n+1\}$ parallel hyperplanes.

Covering by parallel hyperplanes

Suppose $S \subset \mathbb{R}^n$ is a set of k points.

It is not difficult to notice that S can always be covered by no more than $\max\{1, k - n + 1\}$ parallel hyperplanes.

However, while sufficient, is this number necessary?

Covering by parallel hyperplanes

Suppose $S \subset \mathbb{R}^n$ is a set of k points.

It is not difficult to notice that S can always be covered by no more than $\max\{1, k - n + 1\}$ parallel hyperplanes.

However, while sufficient, is this number necessary?

Question 2

Does there exist a set of $k \geq n$ points in $\mathbb{R}^n$ that cannot be covered by fewer than $k - n + 1$ parallel hyperplanes?

Covering by parallel hyperplanes

Suppose $S \subset \mathbb{R}^n$ is a set of k points.

It is not difficult to notice that S can always be covered by no more than $\max\{1, k - n + 1\}$ parallel hyperplanes.

However, while sufficient, is this number necessary?

Question 2

Does there exist a set of $k \geq n$ points in $\mathbb{R}^n$ that cannot be covered by fewer than $k - n + 1$ parallel hyperplanes?

We are specifically interested in this question for sets on the integer lattice grid.

Covering by parallel hyperplanes

Let $T \geq 1$ be an integer and let

$$C_n(T) := \{\boldsymbol{x} \in \mathbb{Z}^n : |\boldsymbol{x}| \leq T\}$$

be the integer cube of sidelength $2T$ centered at the origin in $\mathbb{R}^n$. Let S be a subset of $C_n(T)$.

Covering by parallel hyperplanes

Let $T \geq 1$ be an integer and let

$$C_n(T) := \{\boldsymbol{x} \in \mathbb{Z}^n : |\boldsymbol{x}| \leq T\}$$

be the integer cube of sidelength $2T$ centered at the origin in $\mathbb{R}^n$. Let S be a subset of $C_n(T)$.

Question 3

How many parallel hyperplanes does it take to cover S?

Covering by parallel hyperplanes

Let $T \geq 1$ be an integer and let

$$C_n(T) := \{\mathbf{x} \in \mathbb{Z}^n : |\mathbf{x}| \leq T\}$$

be the integer cube of sidelength $2T$ centered at the origin in $\mathbb{R}^n$. Let S be a subset of $C_n(T)$.

Question 3

How many parallel hyperplanes does it take to cover S?

First notice that $2T + 1$ parallel hyperplanes cover all of $C_n(T)$.

Covering by parallel hyperplanes

Let $T \geq 1$ be an integer and let

$$C_n(T) := \{\boldsymbol{x} \in \mathbb{Z}^n : |\boldsymbol{x}| \leq T\}$$

be the integer cube of sidelength $2T$ centered at the origin in $\mathbb{R}^n$. Let S be a subset of $C_n(T)$.

Question 3

How many parallel hyperplanes does it take to cover S?

First notice that $2T+1$ parallel hyperplanes cover all of $C_n(T)$.

Lemma 5

Let $S \subseteq C_n(T)$ be a set of points of cardinality k.

1. *If no fewer than $k-n+1$ parallel hyperplanes can cover S, then $k \leq 2T+n$.*
2. *If $2T+1$ parallel hyperplanes are required to cover S, then $k \geq 2T+n$.*

Extremal construction

Question 4

Does there exist a subset S of $C_n(T)$ of cardinality $2T + n$ that cannot be covered by fewer than $2T + 1$ parallel hyperplanes?

Extremal construction

Question 4

Does there exist a subset S of $C_n(T)$ of cardinality $2T + n$ that cannot be covered by fewer than $2T + 1$ parallel hyperplanes?

We can answer this question in the affirmative for $T = 1$:

Proposition 6

For each $n \geq 1$ there exist sets $S_n \subset C_n(1)$ of cardinality $n + 2$ which cannot be covered by fewer than 3 parallel hyperplanes.

Extremal construction

Question 4

Does there exist a subset S of $C_n(T)$ of cardinality $2T+n$ that cannot be covered by fewer than $2T+1$ parallel hyperplanes?

We can answer this question in the affirmative for $T=1$:

Proposition 6

For each $n \geq 1$ there exist sets $S_n \subset C_n(1)$ of cardinality $n+2$ which cannot be covered by fewer than 3 parallel hyperplanes.

Example 2

$$S_n = \left\{ \mathbf{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n, -\sum_{i=1}^{n} \boldsymbol{e}_i \right\} \subset \mathbb{R}^n.$$

Extremal construction

Example 3

$$S_n = \left\{ \sum_{j=1}^{n} \boldsymbol{e}_j, \ \sum_{j=n-i+2}^{n} \boldsymbol{e}_j - \boldsymbol{e}_{n-i+1} \ \ \forall\, 1 \le i \le n \right\} \subset \mathbb{R}^n.$$

Extremal construction

Example 3

$$S_n = \left\{ \sum_{j=1}^{n} \mathbf{e}_j,\ \sum_{j=n-i+2}^{n} \mathbf{e}_j - \mathbf{e}_{n-i+1} \ \ \forall\ 1 \le i \le n \right\} \subset \mathbb{R}^n.$$

More generally, one can take any set in $\mathbb{R}^n$ consisting of the origin together with the $n+1$ vertices of any simplex containing the origin in its interior – this will not necessarily be in the $C_n(1)$ grid, but it has the same extremal covering property.

Back to sensing matrices

The extremal sets constructed above can be used to obtain basic sensing matrices as follows.

Proposition 7

Let $k > n$ *and* $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{k-1} \in \mathbb{R}^n$ *be distinct nonzero vectors. Let*

$$S = \{\mathbf{0}, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{k-1}\} \subset \mathbb{R}^n.$$

Let A *be the* $n \times (k-1)$ *matrix, whose columns are these vectors, i.e.*

$$A = \begin{pmatrix} \boldsymbol{x}_1 & \dots & \boldsymbol{x}_{k-1} \end{pmatrix}.$$

If S *cannot be covered by fewer than* $k - n + 1$ *parallel hyperplanes, then* A *is a sensing matrix for n-sparse signals.*

Back to sensing matrices

The extremal sets constructed above can be used to obtain basic sensing matrices as follows.

Proposition 7

Let $k > n$ *and* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{k-1} \in \mathbb{R}^n$ *be distinct nonzero vectors. Let*

$$S = \{\mathbf{0}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{k-1}\} \subset \mathbb{R}^n.$$

Let A *be the* $n \times (k-1)$ *matrix, whose columns are these vectors, i.e.*

$$A = \begin{pmatrix} \boldsymbol{x}_1 & \ldots & \boldsymbol{x}_{k-1} \end{pmatrix}.$$

If S *cannot be covered by fewer than* $k - n + 1$ *parallel hyperplanes, then* A *is a sensing matrix for* n*-sparse signals.*

We can do a lot better with some more work.

Difference sets

Define a partition of $S = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\} \subset \mathbb{R}^n$ into disjoint subsets

$$I_m = \{\boldsymbol{x}_{i_1}, \ldots, \boldsymbol{x}_{i_m}\},\ J_l = \{\boldsymbol{x}_{j_1}, \ldots, \boldsymbol{x}_{j_l}\} = S \setminus I_m,$$

where $m, l \geq 1$ are such that $k = m + l$.

Difference sets

Define a partition of $S = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\} \subset \mathbb{R}^n$ into disjoint subsets

$$I_m = \{\boldsymbol{x}_{i_1}, \ldots, \boldsymbol{x}_{i_m}\},\ J_l = \{\boldsymbol{x}_{j_1}, \ldots, \boldsymbol{x}_{j_l}\} = S \setminus I_m,$$

where $m, l \geq 1$ are such that $k = m + l$.

Define the corresponding set of pairwise difference vectors

$$\mathcal{D}(I_m, J_l) = \{\boldsymbol{x}_i - \boldsymbol{x}_j : \boldsymbol{x}_i \in I_m, \boldsymbol{x}_j \in J_l\},$$

so $|\mathcal{D}(I_m, J_l)| \leq ml = m(k - m)$.

Difference sets

Define a partition of $S = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\} \subset \mathbb{R}^n$ into disjoint subsets

$$I_m = \{\boldsymbol{x}_{i_1}, \ldots, \boldsymbol{x}_{i_m}\},\ J_l = \{\boldsymbol{x}_{j_1}, \ldots, \boldsymbol{x}_{j_l}\} = S \setminus I_m,$$

where $m, l \geq 1$ are such that $k = m + l$.

Define the corresponding set of pairwise difference vectors

$$\mathcal{D}(I_m, J_l) = \{\boldsymbol{x}_i - \boldsymbol{x}_j : \boldsymbol{x}_i \in I_m, \boldsymbol{x}_j \in J_l\},$$

so $|\mathcal{D}(I_m, J_l)| \leq ml = m(k - m)$.

For a subset $D \subseteq \mathcal{D}(I_m, J_l)$ define **support** of D to be the set of all distinct vectors $\boldsymbol{x}_i$ that appear in the differences in D.

Difference sets

Define a partition of $S = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\} \subset \mathbb{R}^n$ into disjoint subsets

$$I_m = \{\boldsymbol{x}_{i_1}, \ldots, \boldsymbol{x}_{i_m}\},\ J_l = \{\boldsymbol{x}_{j_1}, \ldots, \boldsymbol{x}_{j_l}\} = S \setminus I_m,$$

where $m, l \geq 1$ are such that $k = m + l$.

Define the corresponding set of pairwise difference vectors

$$\mathcal{D}(I_m, J_l) = \{\boldsymbol{x}_i - \boldsymbol{x}_j : \boldsymbol{x}_i \in I_m, \boldsymbol{x}_j \in J_l\},$$

so $|\mathcal{D}(I_m, J_l)| \leq ml = m(k - m)$.

For a subset $D \subseteq \mathcal{D}(I_m, J_l)$ define **support** of D to be the set of all distinct vectors $\boldsymbol{x}_i$ that appear in the differences in D. For instance, the support of the difference set

$$\{\boldsymbol{x}_1 - \boldsymbol{x}_2, \boldsymbol{x}_3 - \boldsymbol{x}_2, \boldsymbol{x}_1 - \boldsymbol{x}_4, \boldsymbol{x}_3 - \boldsymbol{x}_4\}$$

is $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4\}$.

Bipartite graphs

Define a bipartite graph $\Gamma(D)$ with vertices corresponding to the support of D. Two vertices $\boldsymbol{x}_i, \boldsymbol{x}_j$ are then connected by an edge if and only if $\boldsymbol{x}_i - \boldsymbol{x}_j \in D$.

Bipartite graphs

Define a bipartite graph $\Gamma(D)$ with vertices corresponding to the support of D. Two vertices $\boldsymbol{x}_i, \boldsymbol{x}_j$ are then connected by an edge if and only if $\boldsymbol{x}_i - \boldsymbol{x}_j \in D$.

In other words D is the set of edges of $\Gamma(D)$. We write $g(D)$ for the minimal length of a cycle in the graph $\Gamma(D)$ called the **girth** of this graph.

We write $A(D)$ for the matrix whose column vectors are elements of the set D.

Better sensing matrices

Theorem 8 (F., Hsu – 2019)

Let $S = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\} \subset \mathbb{R}^n$ be a collection of $k > n$ points, $m, l \geq 1$ integers such that $k = m + l$, $S = I_m \sqcup J_l$ partition of S, and $D \subseteq \mathcal{D}(I_m, J_l)$. Let $1 \leq \ell \leq n - 1$. The following two statements are true:

1. *If S cannot be covered by fewer than $k - n + 1$ parallel hyperplanes and for every subset D' of ℓ vectors of D, $g(D') > \ell$, then $A(D)$ is a sensing matrix for ℓ-sparse vectors.*
2. *If for every $m + l = k$ and partition $S = I_m \sqcup J_l$, $A(D(I_m, J_l))$ is a sensing matrix for n-sparse vectors, then S cannot be covered by fewer than $k - n + 1$ parallel hyperplanes.*

Proof of Theorem 4

Results of Labeznik, Ustimenko and Woldar (1995) provide a deterministic construction of bipartite graphs of girth $\geq \ell + 1$ on k vertices and the number of edges

$$\geq \left(\frac{k}{2}\right)^{1+\frac{2}{3\ell-2}}. \tag{1}$$

Proof of Theorem 4

Results of Labeznik, Ustimenko and Woldar (1995) provide a deterministic construction of bipartite graphs of girth $\geq \ell + 1$ on k vertices and the number of edges

$$\geq \left(\frac{k}{2}\right)^{1+\frac{2}{3\ell-2}}. \quad (1)$$

Let S_n be the set of $n+2$ vectors with $\{0, \pm 1\}$ coordinates obtained in Proposition 6, hence S_n cannot be covered by $(n+2) - n + 1 = 3$ parallel hyperplanes.

Proof of Theorem 4

Results of Labeznik, Ustimenko and Woldar (1995) provide a deterministic construction of bipartite graphs of girth $\geq \ell + 1$ on k vertices and the number of edges

$$\geq \left(\frac{k}{2}\right)^{1+\frac{2}{3\ell-2}}. \qquad (1)$$

Let S_n be the set of $n+2$ vectors with $\{0, \pm 1\}$ coordinates obtained in Proposition 6, hence S_n cannot be covered by $(n+2) - n + 1 = 3$ parallel hyperplanes.

Let Γ be a bipartite graph on the $n+2$ vertices corresponding to the vectors of S_n with the number of edges satisfying (1).

Proof of Theorem 4

Results of Labeznik, Ustimenko and Woldar (1995) provide a deterministic construction of bipartite graphs of girth $\geq \ell + 1$ on k vertices and the number of edges

$$\geq \left(\frac{k}{2}\right)^{1+\frac{2}{3\ell-2}}. \tag{1}$$

Let S_n be the set of $n+2$ vectors with $\{0, \pm 1\}$ coordinates obtained in Proposition 6, hence S_n cannot be covered by $(n+2)-n+1=3$ parallel hyperplanes.

Let Γ be a bipartite graph on the $n+2$ vertices corresponding to the vectors of S_n with the number of edges satisfying (1).

Let D be the set of difference vectors corresponding to the edges of Γ, then $g(D) > \ell$.

Proof of Theorem 4

Therefore by Theorem 8, $A(D)$ is a sensing matrix for ℓ-sparse vectors, and we have $|A(D)| = 2$.

Proof of Theorem 4

Therefore by Theorem 8, $A(D)$ is a sensing matrix for ℓ-sparse vectors, and we have $|A(D)| = 2$.

Furthermore, $A(D)$ is an $n \times d$ integer matrix where

$$d \geq \left(\frac{n+2}{2}\right)^{1+\frac{2}{3\ell-2}},$$

by (1).

Proof of Theorem 4

Therefore by Theorem 8, $A(D)$ is a sensing matrix for ℓ-sparse vectors, and we have $|A(D)| = 2$.

Furthermore, $A(D)$ is an $n \times d$ integer matrix where

$$d \geq \left(\frac{n+2}{2}\right)^{1+\frac{2}{3\ell-2}},$$

by (1).

Notice that if $\ell = o(\log n)$, then $d/n \to \infty$ as $n \to \infty$, meaning that d is bigger than linear in n.

Example

Example 1

Consider the set S_3 *as given in Example 3. Partitioning it into the first three vectors and the remaining two, compute the difference set* D *corresponding to the complete* $(3, 2)$*-bipartite graph* Γ*. Then*

$$A(D) = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 & -2 & -2 \\ -1 & -1 & -2 & -2 & 0 & 0 \end{pmatrix}$$

is a 3×6 *sensing matrix for* 3*-sparse vectors, since* Γ *does not have any* 3*-cycles.*

Tarski planks

We also discuss a curious connection of our covering construction to a simple version of Tarski's plank problem.

Tarski planks

We also discuss a curious connection of our covering construction to a simple version of Tarski's plank problem.

Let M be a nonempty convex compact set in $\mathbb{R}^n$. Its **width** $w(M)$ is defined to be the smallest distance between two parallel supporting hyperplanes to M.

Tarski planks

We also discuss a curious connection of our covering construction to a simple version of Tarski's plank problem.

Let M be a nonempty convex compact set in $\mathbb{R}^n$. Its **width** $w(M)$ is defined to be the smallest distance between two parallel supporting hyperplanes to M.

A **plank** P in $\mathbb{R}^n$ is a strip of space between two parallel hyperplanes and its width $h(P)$ is the distance between them.

Tarski planks

We also discuss a curious connection of our covering construction to a simple version of Tarski's plank problem.

Let M be a nonempty convex compact set in $\mathbb{R}^n$. Its **width** $w(M)$ is defined to be the smallest distance between two parallel supporting hyperplanes to M.

A **plank** P in $\mathbb{R}^n$ is a strip of space between two parallel hyperplanes and its width $h(P)$ is the distance between them.

The classical conjecture of Tarski (1932), proved by Bang (1951) asserts that if a finite collection of planks $P_1, \ldots, P_k$ covers M then

$$\sum_{i=1}^{k} h(P_i) \geq w(M).$$

Tarski planks

The simplest such covering is by parallel planks. But what if we want to cover M by a collection of parallel planks which misses a prescribed collection of points inside of M: how wide can such planks be?

Tarski planks

The simplest such covering is by parallel planks. But what if we want to cover M by a collection of parallel planks which misses a prescribed collection of points inside of M: how wide can such planks be?

Proposition 9

Let M be a compact convex set of width w in $\mathbb{R}^n$ and let $\mathbf{x}_1, \ldots, \mathbf{x}_k$ be points in M. Let S be a collection of all planks P in $\mathbb{R}^n$ with both bounding hyperplanes intersecting M so that the interior of P does not contain any point $\mathbf{x}_i$. For each such plank write $h(P)$ for its width, and define $H := \sup_{P \in S} h(P)$. Then

$$H \geq \begin{cases} \frac{w}{k-n+2} & \text{if } k \geq n \\ \frac{w}{2} & \text{if } k < n. \end{cases} \tag{2}$$

Tarski planks

The bound of Proposition 9 is optimal. Take, for instance, $n = 2$ and let M be an equilateral triangle with height equal to 2, then $w(M) = 2$. Let $k = 3$ and $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in M$ be vertices of a scaled-down equilateral triangle $M' = \frac{1}{3}M$ with height equal to $2/3$ centered at the center of M.

Tarski planks

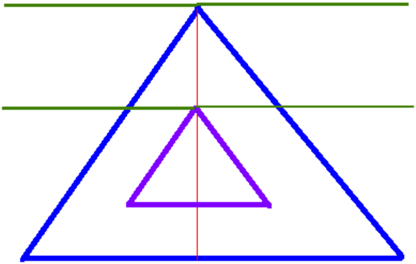
The bound of Proposition 9 is optimal. Take, for instance, $n = 2$ and let M be an equilateral triangle with height equal to 2, then $w(M) = 2$. Let $k = 3$ and $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in M$ be vertices of a scaled-down equilateral triangle $M' = \frac{1}{3}M$ with height equal to $2/3$ centered at the center of M.

The set of planks S as in the statement of Proposition 9 contains a plank P whose one boundary line passes through vertices $\boldsymbol{x}_1, \boldsymbol{x}_2$ and the other through $\boldsymbol{x}_3$. Then

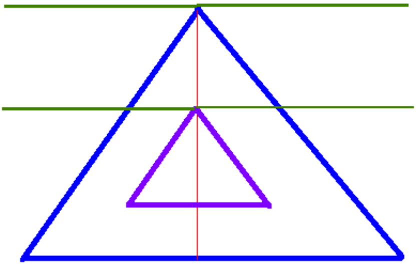
$$h(P) = w(M') = \frac{2}{3} = \frac{w(M)}{k - n + 2},$$

which is precisely the lower bound of the proposition. Notice that no plank in S can have width greater than $h(P)$, and thus the lower bound of (2) when $k \geq n$ is achieved.

Tarski planks



Tarski planks

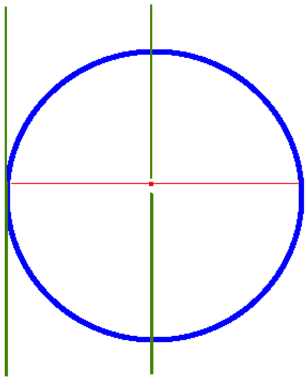


In the situation $k < n$, we can take $n = 2$ again, M a unit disk, $k = 1$ and $\boldsymbol{x}_1 =$ center of M. Then a plank P bounded by a line through $\boldsymbol{x}_1$ and a parallel line tangent to the boundary circle of M has maximal possible width of all planks in S, and

$$h(P) = 1 = \frac{w(M)}{2},$$

the lower bound of (2) in this case.

Tarski planks



References

L. Fukshansky, D. Needell, B. Sudakov, *An algebraic perspective on integer sparse recovery*, Applied Mathematics and Computation, vol. 340 (2019), pg. 31–42

L. Fukshansky, A. Hsu, *Covering point-sets with parallel hyperplanes and sparse signal recovery*, to appear in Discrete and Computational Geometry

Preprints are available at:

`http://math.cmc.edu/lenny/research.html`

References

L. Fukshansky, D. Needell, B. Sudakov, *An algebraic perspective on integer sparse recovery*, Applied Mathematics and Computation, vol. 340 (2019), pg. 31–42

L. Fukshansky, A. Hsu, *Covering point-sets with parallel hyperplanes and sparse signal recovery*, to appear in Discrete and Computational Geometry

Preprints are available at:

`http://math.cmc.edu/lenny/research.html`

Thank you!