On the geometry of period lattices for elliptic curves

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Similarity classes of planar lattices

Every $A \in \mathrm{GL}_2(\mathbb{R})$ is a basis matrix for some planar lattice

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Every lattice $\Omega \in \mathcal{L}_2$ is similar to a unique lattice of the form

$$\Gamma_\tau := \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \mathbb{Z}^2 \text{ for some } \tau := a + bi \text{ in}$$

$$\mathcal{F} := \{\tau = a + bi \in \mathbb{C} : 0 \leq a \leq 1/2, b \geq 0, |\tau| \geq 1\}.$$

We refer to $\mathcal{F}$ as the **set of similarity classes** of lattices in $\mathcal{L}_2$.

Elliptic curves and isogenies

Given lattices $\Lambda, \Lambda' \subset \mathbb{C}$ a nonzero morphism $\mathcal{E} \to \mathcal{E}'$ between the corresponding elliptic curves $\mathcal{E} = \mathbb{C}/\Lambda$ and $\mathcal{E}' = \mathbb{C}/\Lambda'$ which takes 0 to 0 is called an **isogeny**.

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$$\deg(\beta) = [\Lambda' : \beta\Lambda].$$

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This is precisely the size of its kernel. If an isogeny $\mathcal{E} \to \mathcal{E}'$ exists, then there also exists the dual isogeny $\mathcal{E}' \to \mathcal{E}$ of the same degree such that their composition is the multiplication-by-degree map, and hence the curves are called **isogenous**: this is an equivalence relation. There may exist multiple isogenies between two elliptic curves, but since degree of an isogeny is a positive integer, we can ask for an isogeny of minimal degree.

Isomorphism classes of elliptic curves

An **isomorphism** of elliptic curves is an injective isogeny, i.e. of degree one. Each elliptic curve is isomorphic to an elliptic curve $\mathcal{E}_\tau$ with period lattice

$$\Gamma_\tau = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \mathbb{Z}^2 \text{ for some } \tau := a + bi \text{ in}$$

$$\mathcal{D} := \{\tau = a + bi \in \mathbb{C} : -1/2 < a \leq 1/2, b \geq 0, |\tau| \geq 1\}.$$

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This set $\mathcal{D}$ can also be viewed as a fundamental domain for the action of the group $\mathrm{SL}_2(\mathbb{Z})$ on the set of lattices Γ_τ by right matrix multiplication by g^{-1} for each $g \in SL_2(\mathbb{Z})$:

$$\Gamma_\tau = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \mathbb{Z}^2 \mapsto g \cdot \Gamma_\tau := \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} g^{-1} \mathbb{Z}^2.$$

Arithmetic, well-rounded, semi-stable lattices

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Successive minima of Γ are real numbers $0 < \lambda_1(\Gamma) \leq \lambda_2(\Gamma)$:

$$\lambda_i(\Gamma) := \min\left\{r \in \mathbb{R}_{>0} : \dim_{\mathbb{R}} \operatorname{span}_{\mathbb{R}}\left(\mathbb{B}(r) \cap L\right) \geq i\right\},$$

where $\mathbb{B}(r)$ is the disk of radius r centered at the origin in $\mathbb{R}^2$. Γ is called **well-rounded (WR)** if $\lambda_1(\Gamma) = \lambda_2(\Gamma)$. WR lattices are central to discrete optimization and connected areas.

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Γ is called **semi-stable** if

$$\lambda_1(L)^2 \geq \det(\Gamma) := |\det(A)|.$$

Semi-stable lattices are important in reduction theory of algebraic groups.

Geometrically speaking...

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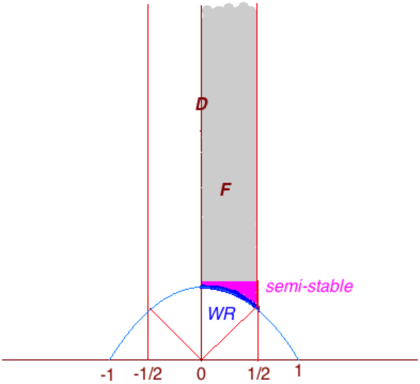

Algebraically speaking...

Γ_τ is arithmetic iff $\tau \in \mathcal{F}$ is of the form

$$\tau = \tau(a,b,c,d) := \frac{a}{b} + i\sqrt{\frac{c}{d}}$$

for some integers a, b, c, d such that

$$\gcd(a,b) = \gcd(c,d) = 1,\ 0 \leq a \leq b/2,\ c/d \geq 1 - a^2/b^2.$$

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This condition is equivalent to the elliptic curve $\mathcal{E}_\tau$ with period lattice Γ_τ having **complex multiplication** (CM) by the imaginary quadratic field $\mathbb{Q}(\tau)$, i.e. the endomorphism ring of $\mathcal{E}_\tau$ is an order in $\mathbb{Q}(\tau)$ properly containing $\mathbb{Z}$.

The j-invariant

The **Klein j-function** is a bijective holomorphic map $j : \mathcal{D}' \to \mathbb{C}$. If $\mathcal{E}$ is an elliptic curve, then it is isomorphic to an elliptic curve $\mathcal{E}_\tau$ for precisely one $\tau \in \mathcal{D}'$, and hence the value $j(\tau)$ is an invariant of this elliptic curve, called its **j-invariant**.

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- For $\tau \in \mathcal{D}$, $j(\tau) \in \mathbb{R}$ iff τ belongs to the boundary of $\mathcal{F}$, and Γ_τ is WR iff $j(\tau) \in [0, 1]$.

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- For $\tau \in \mathcal{D}$, $j(\tau) \in \mathbb{R}$ iff τ belongs to the boundary of $\mathcal{F}$, and Γ_τ is WR iff $j(\tau) \in [0, 1]$.
- Suppose $\tau \in \mathcal{F}$ is algebraic. Then

$$\Gamma_\tau \text{ is arithmetic } \iff \deg_{\mathbb{Q}}(\tau) = 2 \iff j(\tau) \in \overline{\mathbb{Q}}.$$

  In this case, the degree of the algebraic number $j(\tau)$ is the class number of the quadratic imaginary number field $\mathbb{Q}(\tau)$.

Maximum height

We define the **maximum height** of an arithmetic similarity class $\Gamma_{\tau(a,b,c,d)}$ to be

$$\mathfrak{m}(\Gamma_{\tau(a,b,c,d)}) = \mathfrak{m}(\tau(a,b,c,d)) := \max\{|a|, |b|, |c|, |d|\}.$$

Then, the number of arithmetic similarity classes with $\mathfrak{m}(\Gamma_{\tau(a,b,c,d)}) \leq T$ is finite for every real number T.

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Then, the number of arithmetic similarity classes with $\mathfrak{m}(\Gamma_{\tau(a,b,c,d)}) \leq T$ is finite for every real number T.

Let $T \in \mathbb{Z}_{>0}$, and define

$$\begin{aligned} N_1(T) &= \left|\{\Lambda_\tau : \Lambda_\tau \text{ is arithmetic and } \mathfrak{m}(\Lambda_\tau) \leq T\}\right|, \\ N_2(T) &= \left|\{\Lambda_\tau : \Lambda_\tau \text{ is arithmetic semi-stable and } \mathfrak{m}(\Lambda_\tau) \leq T\}\right|, \\ N_3(T) &= \left|\{\Lambda_\tau : \Lambda_\tau \text{ is arithmetic WR and } \mathfrak{m}(\Lambda_\tau) \leq T\}\right|. \end{aligned}$$

Counting estimate

Theorem 1 (F., Guerzhoy, Luca (2015))

With notation as above, $N_1(T) > N_2(T) > N_3(T)$*, and as* $T \to \infty$,

$$N_1(T) = \frac{39T^4}{8\pi^4} + O(T^3 \log T),$$

and

$$N_2(T) = \frac{3T^4}{8\pi^4} + O(T^3 \log T),$$

while

$$N_3(T) = \frac{3T^2}{2\pi^2} + O(T \log T).$$

Consequence

It follows from Theorem 1 that about 7.7% of arithmetic similarity classes in the plane are semi-stable, when counted with respect to the maximum height.

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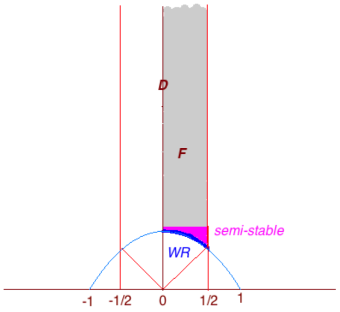

On the other hand, only about 4.5% (with respect to Poincare measure) of all similarity classes in the plane are semi-stable.

Virtually rectangular lattices

A planar lattice M is **rectangular** if it has an orthogonal basis, i.e. if it is similar to a lattice of the form

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L is **virtually rectangular** if it contains a rectangular sublattice M of finite index. If M is a rectangular sublattice of L of smallest index, then L is similar to L' containing a sublattice of the form M' as in (1) similar to M and

$$[L : M] = [L' : M'] = \frac{\det(L)}{|ab|}.$$

Virtually rectangular elliptic curves

Theorem 2 (F., Guerzhoy, Kühnlein (2020))

Let $\tau = a + bi \in \mathcal{D}$ and let E_τ be the corresponding elliptic curve with the period lattice Γ_τ. The following are equivalent:

1. *Either $a \in \mathbb{Q}$ or there exists some $t \in \mathbb{R}$ such that $\frac{v'}{v} := a - bt, \frac{w'}{w} := a + b/t \in \mathbb{Q}$,*
2. *Γ_τ is virtually rectangular,*
3. *E_τ is isogenous to E' with real j-invariant ≥ 1,*
4. *E_τ is isogenous to E' with real j-invariant in $[0, 1]$.*

If (1) - (4) hold with $a = p/q \in \mathbb{Q}$, then exists an isogeny $E' \to E_\tau$ with degree $\delta(E'/E_\tau) = q$. If If (1) - (4) hold with $a \notin \mathbb{Q}$ and $t \in \mathbb{R}$ satisfies (1), then exists an isogeny $E' \to E_\tau$ with

$$\delta(E'/E_\tau) = \frac{|b||vw|(t^2+1)}{|t|}.$$

Remarks

- Degree of the isogeny is precisely the smallest index of a rectangular sublattice.

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- CM elliptic curves are the only ones whose period lattice contains non-parallel rectangular sublattices: in the CM case, there are infinitely many t satisfying condition (1) of Theorem 2 (each corresponding to a different rectangular sublattice), whereas for all other virtually rectangular elliptic curves such t is essentially unique.

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- CM elliptic curves are the only ones whose period lattice contains non-parallel rectangular sublattices: in the CM case, there are infinitely many t satisfying condition (1) of Theorem 2 (each corresponding to a different rectangular sublattice), whereas for all other virtually rectangular elliptic curves such t is essentially unique.
- Virtually rectangular lattices in the plane have intrinsic geometric meaning in terms of the corresponding points on the modular curve: they correspond precisely to the points that lie on geodesics intersecting the real axis at rational points only.

Deep holes

Let $L \subset \mathbb{R}^2$ be a lattice with successive minima $\lambda_1 \leq \lambda_2$ and the corresponding minimal basis vectors $\boldsymbol{x}_1, \boldsymbol{x}_2$. It is well known that, choosing $\pm\boldsymbol{x}_1, \pm\boldsymbol{x}_2$ if necessary, we can ensure that the angle θ between these vectors is in the interval $[\pi/3, \pi/2]$: this angle is an invariant of the lattice, we call it **angle** of L.

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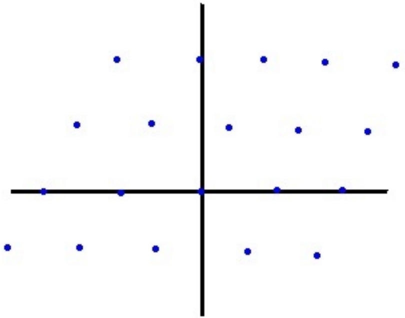
A **deep hole** of L is a point in $\mathbb{R}^2$ which is farthest away from the lattice. The distance from the origin to the nearest deep hole is the *covering radius* μ of L. There is a unique deep hole $\boldsymbol{z}$ of L contained in the triangle T with vertices $\boldsymbol{0}$ and the endpoints of $\boldsymbol{x}_1, \boldsymbol{x}_2$: we call it the *fundamental deep hole* of L. Define the **deep hole lattice** of L to be

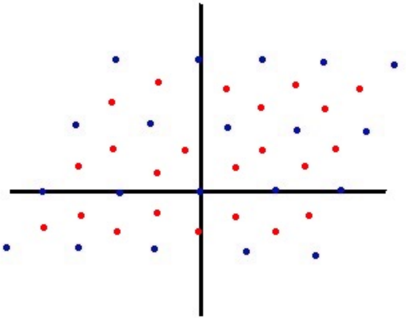
$$H(L) := \operatorname{span}_{\mathbb{Z}}\{\boldsymbol{x}_1, \boldsymbol{z}\}.$$

Deep holes



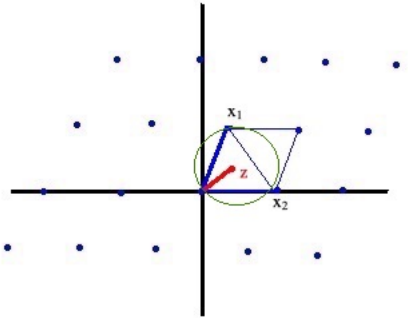
Lattice points in blue

Deep holes



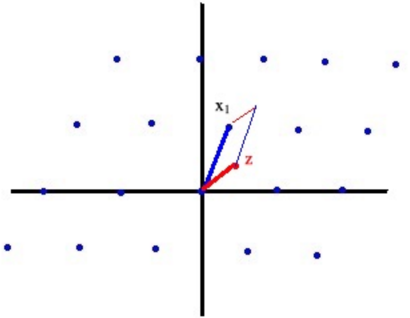
Deep holes in red

Deep holes



Fundamental deep hole

Deep holes



Deep hole lattice: $H(L) = \text{span}_{\mathbb{Z}}\{\boldsymbol{x}_1, \boldsymbol{z}\}$.

Properties of deep hole lattices

Theorem 3 (F., Guerzhoy, Nielsen (2023))

Let L be a lattice in the plane with the angle $\theta \in [\pi/3, \pi/2]$ and successive minima λ_1 and $\lambda_2 = \alpha\lambda_1$ for some $\alpha \geq 1$. Let $H(L)$ be the deep hole lattice of L. The following statements hold:

1. *If $\alpha \leq 2\sin(\theta + \pi/6)$, then $H(L)$ is WR.*
2. *If L is semi-stable, then $H(L)$ is WR.*
3. *If L is WR, then $H(L) \sim L$.*
4. *If $L \subset K^2$ for some subfield K of $\mathbb{R}$, then $H(L) \subset K^2$.*

Deep hole lattices in the fundamental strip

Next we turn our attention specifically to the lattices of the form Γ_τ for $\tau \in \mathcal{F}$ parameterizing all the similarity classes in the plane. Given a subfield K of $\mathbb{R}$, we say that a similarity class represented by τ **lies over** K if $\tau = a + bi$ with real numbers $a, b \in K$. This is equivalent to saying that some lattice in this similarity class is contained in $K(i) \subseteq \mathbb{C}$, which is identified with $K^2 \subseteq \mathbb{R}^2$.

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Theorem 4 (F., Guerzhoy, Nielsen (2023))

Let $\tau_0 = a_0 + b_0 i \in \mathcal{F}$ with $a_0, b_0 \in K$ for some subfield $K \subseteq \mathbb{R}$. There exists a finite sequence of numbers $\tau_1, \dots, \tau_n$ given by $\tau_k = a_k + b_k i$ for all $1 \leq k \leq n$, so that

$$a_k = \frac{1}{2},\ b_k = \frac{a_{k-1}^2 + b_{k-1}^2 - a_{k-1}}{2b_{k-1}} \in K\ \forall\ 1 \leq k \leq n, \qquad (2)$$

with $\Gamma_{\tau_k} = H(\Gamma_{\tau_{k-1}})$ and Γ_{τ_n} WR, hence $H(\Gamma_{\tau_n}) \sim \Gamma_{\tau_n}$. Also, $\tau_1, \dots, \tau_{n-1} \in \mathcal{F}$, $|\tau_n| \leq 1$ and $n \leq \log_2\left(2b_0/\sqrt{3}\right)$.

Deep hole sequence

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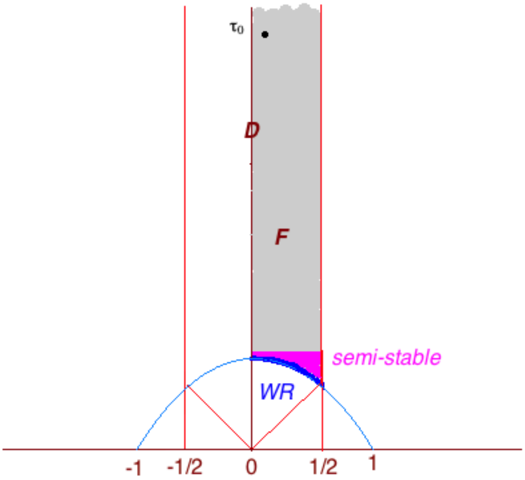

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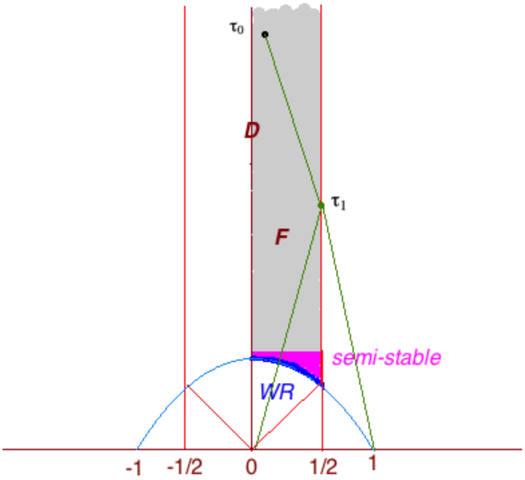

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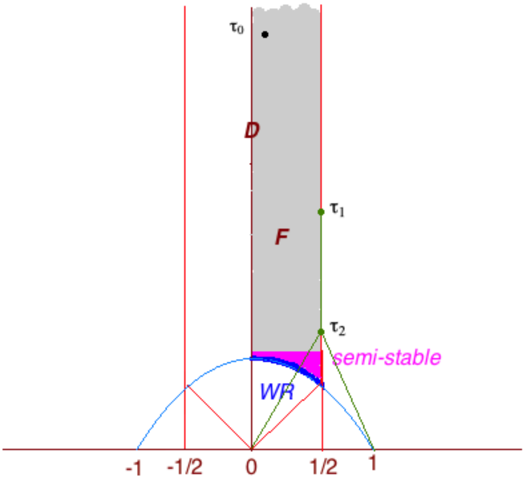

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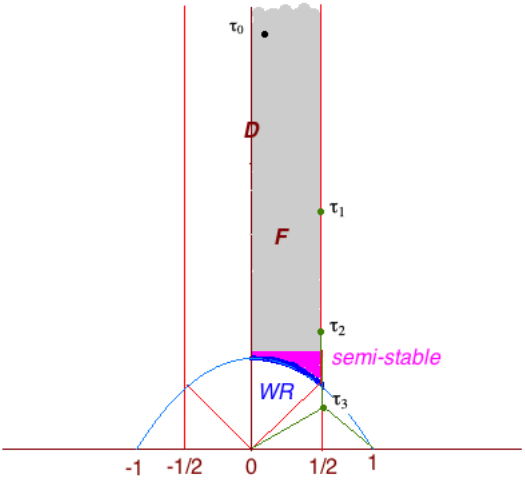

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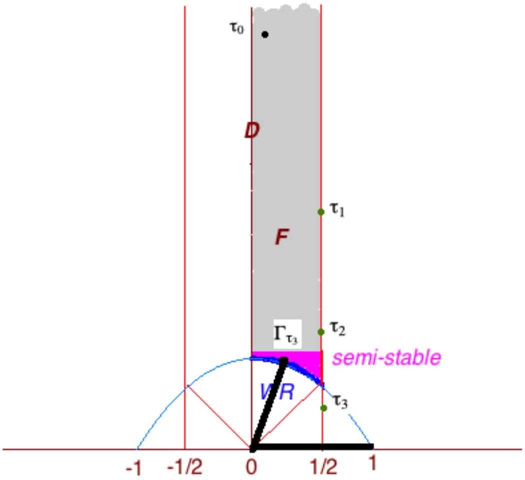

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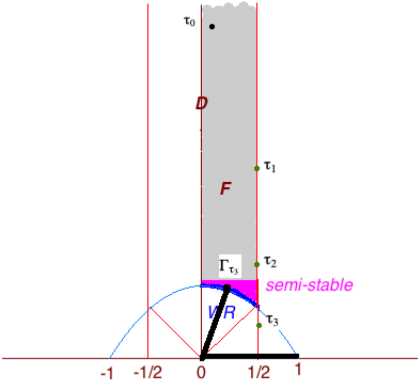

Thus, the map $\tau_i \to \tau_{i+1}$ defines a dynamical system, in which every point is (pre-)periodic with orbit size n as in Theorem 4.

CM case

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Theorem 5 (F., Guerzhoy, Nielsen (2023))

Let $\tau_0 = a_0 + b_0 i \in \mathcal{F}$ *be a quadratic irrationality and*

$$\{\tau_k = a_k + b_k i\}_{k=1}^{n}$$

its corresponding deep hole sequence. For each $0 \leq k \leq n$, *let* $\mathcal{E}_{\tau_k}$ *be the corresponding CM elliptic curve with the arithmetic period lattice* Γ_{τ_k}. *Then all of these elliptic curves are isogenous. Furthermore, for any* $0 \leq k \leq n-1$, *there exists an isogeny between* $\mathcal{E}_{\tau_k}$ *and* $\mathcal{E}_{\tau_{k+1}}$ *with degree*

$$\delta_k \leq \frac{12\sqrt{3}\ b_{k+1}\ d_k^4\ (a_k^2 + b_k^2)^2}{b_k},$$

where $d_k = \min\{d \in \mathbb{Z}_{>0} : da_k, d^2 b_k^2 \in \mathbb{Z}\}$.

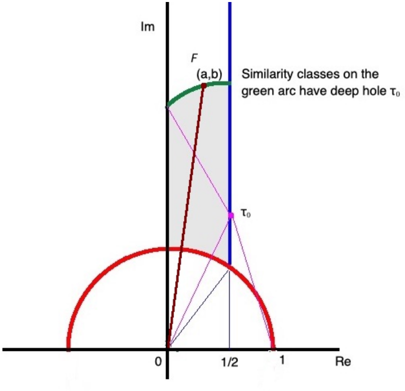
Lattices with a prescribed deep hole

Now, we consider a certain inverse problem. Let K be a number field of degree n and suppose that the similarity class represented by $\tau_0 = \frac{1}{2} + it \in \mathcal{F}$ lies over K. Consider the set

$$S_{K,\tau_0} = \{\tau \in \mathcal{F} : \tau \text{ is defined over } K \text{ and } H(\Gamma_\tau) = \Gamma_{\tau_0}\}. \qquad (3)$$

i.e., the set of similarity classes defined over K whose deep hole lattice is Γ_{τ_0}. While this is an infinite set, we can count these similarity classes bounding the so-called primitive height $\mathcal{H}^p$ of τ.

Lattices with a prescribed deep hole



Similarity classes with a prescribed deep hole. Pink lines are radii of the circle centered at τ_0. The brown line $y = \frac{b}{a}x$ intersects the green arc at a point $\tau = a + bi$ defined over K.

The primitive height

- $\Delta_K =$ be the discriminant of K
- $\mathcal{O}_K =$ ring of integers of K
- $r_1 =$ number of real embeddings, $r_2 =$ number of conjugate pairs of complex embeddings, so $n = r_1 + 2r_2$
- $M(K) =$ set of place of K

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- $M(K)$ = set of place of K

For a point $\boldsymbol{x} \in K^m$, define its **denominator** to be

$$d(\boldsymbol{x}) = \min\{c \in \mathbb{Q}_{>0} : c\boldsymbol{x} \in \mathcal{O}_K^m\}, \tag{4}$$

and let the **(rationally) primitive point** corresponding to $\boldsymbol{x}$ be $\boldsymbol{x}_p = d(\boldsymbol{x})\boldsymbol{x}$.

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and let the **(rationally) primitive point** corresponding to $\boldsymbol{x}$ be $\boldsymbol{x}_p = d(\boldsymbol{x})\boldsymbol{x}$.
We define the **primitive height** of $\boldsymbol{x} \in K^m$ to be

$$\mathcal{H}^p(\boldsymbol{x}) := \max_{v|\infty} |\boldsymbol{x}_p|_v.$$

The counting estimate

Theorem 6 (F., Guerzhoy, Nielsen (2023))

For a real number $T \geq 1$, define

$$S_{K,\tau_0}(T) = \{\tau \in S_{K,\tau_0} : \mathcal{H}^p(\tau) \leq T\},$$

where $\tau_0 = \frac{1}{2} + it \in \mathcal{F}$ lies over K. Then, as $T \to \infty$,

$$|S_{K,\tau_0}(T)| \leq \left(\frac{4^{r_1}\pi^{2r_2}}{8\zeta(2n)\left(2t + \sqrt{4t^2+1}\right)|\Delta_K|} \right) T^{2n} + O(T^{2n-1}),$$

where ζ stands for the Riemann zeta-function and $n = [K : \mathbb{Q}]$.

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Thank you!