On a new version of Siegel's lemma

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Thue & Siegel

The following observation was first made by A. Thue (1909) and then formally proved by C. L. Siegel (1929).

Theorem 1 (Siegel's Lemma)

Let A be a nonzero integer $m \times n$ matrix, $1 \leqslant m < n$. Then there exits $\mathbf{0} \neq \boldsymbol{x} \in \mathbb{Z}^n$ such that $A\boldsymbol{x} = \mathbf{0}$ and

$$|\boldsymbol{x}| \leqslant 1 + (n|A|)^{\frac{m}{n-m}}, \tag{1}$$

where $|\cdot|$ is the sup-norm (maximum of the absolute values of the coordinates) of the vector $\boldsymbol{x}$ and the matrix A, respectively.

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where $|\cdot|$ is the sup-norm (maximum of the absolute values of the coordinates) of the vector $\boldsymbol{x}$ and the matrix A, respectively.

This result is used in transcendental number theory and Diophantine approximation for construction of auxiliary polynomials with small integer coefficients which vanish with prescribed multiplicity at given algebraic points. The exponent $\frac{m}{n-m}$ in (2) is sharp.

Invariant version

The bound (2) lacks invariance under linear transformations: for any $m \times m$ integer matrix U,

$$(UA)\boldsymbol{x} = A\boldsymbol{x} = \mathbf{0},$$

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$$(UA)\boldsymbol{x} = A\boldsymbol{x} = \mathbf{0},$$

however $|UA|$ and $|A|$ can be very different. The first *invariant* version of Siegel's lemma was obtained by E. Bombieri and J. Vaaler.

Theorem 2 (Bombieri-Vaaler, 1983)

Let A be a nonzero integer $m \times n$ matrix, $1 \leqslant m < n$. Then there exits $\mathbf{0} \neq \boldsymbol{x} \in \mathbb{Z}^n$ such that $A\boldsymbol{x} = \mathbf{0}$ and

$$|\boldsymbol{x}| \leqslant \left(D^{-1} \sqrt{\det(A^\top A)} \right)^{\frac{1}{n-m}}, \qquad (2)$$

where $D = \gcd$ of the determinants of all $m \times m$ minors of A.

Absolute values

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K = number field, $M(K)$ = its set of places, Δ_K = its discriminant

$d = r_1 + 2r_2 = [K : \mathbb{Q}]$, where r_1 = number of real embeddings and r_2 = number of pairs of complex conjugate embeddings of K

$\forall v \in M(K)$, $d_v = [K_v : \mathbb{Q}_v]$, and $|\cdot|_v$ extends the usual archimedean or the usual p-adic absolute value on $\mathbb{Q}$

Product Formula: $\prod_{v \in M(K)} |a|_v^{d_v} = 1, \ \forall \ 0 \neq a \in K$

Height function

Let $n \geqslant 2$, and define *local norms*

$$|\boldsymbol{x}|_v = \max_{1 \leqslant i \leqslant n} |x_i|_v \ \forall\ v \in M(K), \ \ \|\boldsymbol{x}\|_v = \left(\sum_{i=1}^{n} |x_i|_v^2 \right)^{1/2} \ \forall\ v \mid \infty,$$

where $\boldsymbol{x} = (x_1, \ldots, x_n) \in K^n$.

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where $\boldsymbol{x} = (x_1, \ldots, x_n) \in K^n$.
A **height function** $H : K^n \to \mathbb{R}_{\geqslant 0}$ is then given by

$$H(\boldsymbol{x}) = \left(\prod_{v \nmid \infty} |\boldsymbol{x}|_v^{d_v} \times \prod_{v \mid \infty} \|\boldsymbol{x}\|_v^{d_v} \right)^{1/d}.$$

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By product formula, $H(a\boldsymbol{x}) = H(\boldsymbol{x})$ for every $0 \neq a \in K$, hence H is *projectively defined*. Further, H is *absolute*, i.e. $H(\boldsymbol{x})$ is the same computed over any number field containing coordinates of $\boldsymbol{x}$. We define $H(\mathbf{0}) = 0$.

Schmidt's height on subspaces

We can also talk about height of subspaces of K^n, as first introduced by **W. M. Schmidt** (1967). Let $V \subseteq K^n$ be an m-dimensional subspace, and let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$ be a basis for V.

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Write $\wedge$ for the usual wedge product of vectors, and let

$$\boldsymbol{y} := \boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_m \in K^{\binom{n}{m}}$$

under the standard lexicographic embedding.

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Duality: If $A = (\boldsymbol{a}_1 \dots \boldsymbol{a}_{n-m})^\top$ is an $(n-m) \times n$ matrix over K such that

$$V = \{\boldsymbol{x} \in K^n : A\boldsymbol{x} = \boldsymbol{0}\},$$

then

$$H(V) = H(A) := H(\boldsymbol{a}_1 \wedge \cdots \wedge \boldsymbol{a}_{n-m}).$$

Finiteness property

An important property that height functions satisfy, by analogy with $|\cdot|$ or $\|\cdot\|$ over $\mathbb{Z}$ is *finiteness*.

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Northcott's theorem: *For every* $d, B \in \mathbb{R}_{>0}$ *the set*

$$\left\{[\boldsymbol{x}] \in \mathbb{P}(\overline{\mathbb{Q}}^n) : \deg_{\mathbb{Q}}(\boldsymbol{x}) \leqslant d, H(\boldsymbol{x}) \leqslant B\right\}$$

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More generally, height measures *arithmetic complexity* (by analogy with *degree* in algebraic geometry measuring *geometric complexity*), and so a point of relatively small height is *arithmetically simple*.

Bombieri – Vaaler

The following generalized and powerful *subspace version* of Siegel's lemma was proved by Bombieri and Vaaler.

Theorem 3 (Bombieri-Vaaler, 1983)

Let K be a number field of degree d with r_2 pairs of complex conjugate embeddings and discriminant Δ_K, and let $V \subseteq K^n$ be an m-dimensional subspace, $1 \leqslant m \leqslant n$. There exists a basis $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$ for V over K such that

$$\prod_{i=1}^{m} H(\boldsymbol{x}_i) \leqslant n^{m/2} \left(\left(\frac{2}{\pi} \right)^{r_2} |\Delta_K| \right)^{\frac{m}{2d}} H(V). \tag{3}$$

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In situations when $|\Delta_K|$ is very large, it may overpower $H(V)$ in the upper bound. This calls for an *absolute* version that would not depend on a number field.

Roy – Thunder

An absolute version of Siegel's lemma was established by Roy & Thunder (a similar result was independently obtained by S. Zhang (1995)).

Theorem 4 (Roy-Thunder, 1996)

Let $V \subseteq \overline{\mathbb{Q}}^n$ be an m-dimensional subspace, $1 \leqslant m \leqslant n$. There exists a basis $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$ for V over $\overline{\mathbb{Q}}$ such that

$$\prod_{i=1}^{m} H(\boldsymbol{x}_i) \leqslant \left(e^{\frac{m(m-1)}{4}} + \varepsilon \right) H(V), \tag{4}$$

for any $\varepsilon > 0$ (the choice of the basis depends on ε).

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While the Roy-Thunder bound does not depend on any number field, the vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$ are also not guaranteed to lie over a fixed number field and their degrees over $\mathbb{Q}$ are not bounded.

Monotone basis

A vector is *s-sparse* if it has no more than s nonzero coordinates.

Theorem 5 (Forst-F., 2023)

Let K be a number field and $V = AK^m \subseteq K^n$ be an m-dimensional subspace, $1 \leqslant m < n$, where A is an $n \times m$ basis matrix for V. Let B be a full rank $m \times m$ submatrix of A, and write $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_m\}$ for the column vectors of the matrix AB^{-1}. Then $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$ is another basis for V over K, which consists of $(n - m + 1)$-sparse vectors satisfying the following property: if $I \subseteq \{1, \dots, m\}$ is a nonempty subset, and

$$W_I = \operatorname{span}_K \{\boldsymbol{x}_i : i \in I\},$$

then

$$H(W_I) \leqslant H(V).$$

Monotone basis

Theorem 5, continuation

In particular,

$$\max_{1 \leqslant i \leqslant m} H(\boldsymbol{x}_i) \leqslant H(V). \tag{5}$$

Moreover, if $I_1 \subsetneq I_2 \subseteq \{1, \ldots, m\}$*, then*

$$H(W_{I_1}) \leqslant H(W_{I_2}). \tag{6}$$

Equality is attained in (6) *if and only if* $\boldsymbol{x}_i$ *is a standard basis vector for each* $i \in I_2 \backslash I_1$*.*

Idea of proof

We outline a construction of a basis $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$ for V with

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Since $\dim_K V = m$, there exists an $(n-m) \times n$ matrix A of rank $n-m$ with entries in K so that

$$V = \{\boldsymbol{x} \in K^n : A\boldsymbol{x} = \boldsymbol{0}\},$$

then $H(A) = H(V)$ by the duality principle. Since $\mathrm{rk}(A) = n-m$, there must exist standard basis vectors $\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_m} \in K^n$ so that the matrix $B := \begin{pmatrix} A \\ E \end{pmatrix}$ with $E := (\boldsymbol{e}_{i_1} \ \dots \ \boldsymbol{e}_{i_m})^\top$ is in $\mathrm{GL}_n(K)$.

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For each $1 \leqslant j \leqslant m$ define the vector

$$\boldsymbol{x}_j = B^{-1}\boldsymbol{e}_{j+n-m},$$

which is the $(j+n-m)$-th column vector of the matrix B^{-1}. Then for every $1 \leqslant j \leqslant m$, $A\boldsymbol{x}_j = 0$, i.e. $\boldsymbol{x}_j \in V$.

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Further, these vectors are linearly independent since they are columns of a nonsingular matrix B^{-1}, and so they form a basis for V. We will now estimate their heights.

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Let us write B_j for the $(n-1) \times n$ submatrix of B without the $(j+n-m)$-th row, then $B_j \boldsymbol{x}_j = \mathbf{0}$ since the $(j+n-m)$-th is the only row of B whose dot-product with the $(j+n-m)$-th column of B^{-1} is nonzero.

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Using some standard height inequalities, we obtain:

$$H(\boldsymbol{x}_j) = H(B_j) \leqslant H(A) \prod_{\substack{k=1 \\ k \neq j+n-m}}^{m} H(\boldsymbol{e}_k) = H(A) = H(V),$$

since height of a standard basis vector is equal to 1. This completes the proof.

Some remarks

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- In situations when height of the subspace V is dominated by the constant depending on n and K in Bombieri-Vaaler (3) or on m in Roy-Thunder (4) our bound (5) may be better. Additionally, our bound can be preferable in some applications due to its simplicity.

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- Bombieri-Vaaler and Roy-Thunder arguments rely on sophisticated tools from the geometry of numbers. Ours uses only linear algebra.
- Our result produces a small-height basis with additional *sparse* and *monotone* properties, which can also be valuable in potential applications.

Example

We demonstrate Theorem 5 on a simple example. Let $K = \mathbb{Q}$, $n = 4$, $m = 3$, and take

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 1 \\ 5 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix}.$$

The corresponding vector of Grassmann coordinates is $-18 \cdot (1, 1, 1, 1)$, and hence the height of the subspace $V = A\mathbb{Q}^3 \subset \mathbb{Q}^4$ is $H(V) = 2$. Take the indexing set $I = \{1, 2, 3\}$ and consider the corresponding nonsingular minor

$$A_I = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 1 \\ 5 & 2 & 1 \end{pmatrix}.$$

Example

Then we obtain the new basis matrix

$$X = AA_I^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

The column vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ of X are 2-sparse and of height $\sqrt{2}$ each. Further, the 2-dimensional subspace spanned by any two of these column vectors has height $\sqrt{3}$. For instance, take

$$W_{\{1\}} = \operatorname{span}_{\mathbb{Q}}\{\boldsymbol{x}_1\},\ W_{\{1,2\}} = \operatorname{span}_{\mathbb{Q}}\{\boldsymbol{x}_1, \boldsymbol{x}_2\},$$

and observe that

$$H(W_{\{1\}}) = \sqrt{2} < H(W_{\{1,2\}}) = \sqrt{3} < H(V) = 2.$$

Many bases

We also obtain the following generalization:

Theorem 6 (Forst-F., 2023)

Let $V \subseteq K^n$ be a subspace of dimension m where $1 \leqslant m < n$. For each integer $\ell > m$ there exists a collection of vectors

$$S(\ell) = \{\boldsymbol{y}_1, \dots, \boldsymbol{y}_\ell\} \subset V$$

with the following properties:

1. *Every subset of m vectors from $S(\ell)$ forms a basis for V,*
2. *For every $\boldsymbol{y}_i \in S(\ell)$,*

$$H(\boldsymbol{y}_i) \leqslant m^{3/2} (2\ell)^{\frac{m-1}{m}} \min \left\{ H(V)^m, \gamma_K(m)^{m/2} H(V) \right\},$$

where $\gamma_K(m)^{1/2}$ is the generalized Hermite's constant.

Proof ingredients

The proof of Theorem 6 uses the following ingredients:

- Our Theorem 5
- A version of Bombieri-Vaaler theorem with an improved constant (due to Vaaler, 2003)
- A construction of small-norm integer sensing matrices (due to Konyagin & Sudakov, 2020)

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The idea is to multiply a small-height basis matrix by an integer sensing matrix with controlled coefficients: every maximal minor of such a matrix is nonsingular.

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