Heights and quadratic forms

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Hilbert's Tenth Problem

Consider a system of m Diophantine equations in n variables, i.e.

$$\left.\begin{array}{l} P_1(X_1, \ldots, X_n) = 0 \\ \vdots \\ P_m(X_1, \ldots, X_n) = 0 \end{array}\right\} \tag{1}$$

where $P_1, \ldots, P_m$ are polynomials with integer coefficients.

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The famous result of **Y. Matijasevich** (1970; building on the previous work by **M. Davis, H. Putnam** and **J. Robinson** - 1961) implies that **Question 1 in general is undecidable.**

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If the system (1) *has a nontrivial solution vector* $\boldsymbol{x} \in \mathbb{Z}^n$*, then there exists such a solution vector with*

$$|\boldsymbol{x}| := \max_{1 \leqslant i \leqslant n} |x_i| \leqslant B \tag{2}$$

for some explicit constant $B = B(P_1, \ldots, P_m)$*.*

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for some explicit constant $B = B(P_1, \ldots, P_m)$.

Then to answer Question 1, it would be enough to check whether any of the vectors in the finite set

$$\left\{ \boldsymbol{x} \in \mathbb{Z}^n : \max_{1 \leqslant i \leqslant n} |x_i| \leqslant B \right\}$$

is a solution to (1), reducing it to a **finite search algorithm.**

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Question 3

Assuming the polynomial system $P_1, \dots, P_M$ has a nontrivial integral solution, can we find an explicit search bound?

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This suggests that search bounds for equations of degree $\geqslant 4$ may be out of reach, and relatively little is known even for degree 3 (although some work has been done, especially in the recent years). There is however a wealth of results for degree 1 and 2. This talk will focus specifically on the quadratic case.

Quadratic forms: Cassels' Theorem

Let

$$F(\boldsymbol{X}, \boldsymbol{Y}) = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} X_i Y_j$$

be a symmetric bilinear form in $2n$ variables, $n \geqslant 2$, with integer coefficients, and let $F(\boldsymbol{X}) = F(\boldsymbol{X}, \boldsymbol{X})$ be the associated quadratic form. F is **isotropic** over a set S if it has a nontrivial zero in S.

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Theorem 1 (J. W. S. Cassels - 1955)

If F is isotropic over $\mathbb{Z}$, then there exists $\mathbf{0} \neq \boldsymbol{x} \in \mathbb{Z}^n$ such that $F(\boldsymbol{x}) = 0$ and

$$|\boldsymbol{x}| \ll_n |F|^{\frac{n-1}{2}},$$

where $|F| := \max_{1 \leqslant i,j \leqslant n} |f_{ij}|$ and the constant in the upper bound is explicit.

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The exponent $\frac{n-1}{2}$ in the upper bound is best possible.

The inhomogeneous quadratic case

Now assume that an inhomogeneous quadratic equation in $n \geqslant 3$ variables with integer coefficients

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij}X_iX_j + \sum_{i=1}^{n} f_{i0}X_i + f_{00} = 0$$

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$$|\boldsymbol{x}| \ll_n |F|^{p(n)}, \tag{3}$$

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In case $n = 2$, **Kornhauser** (1990) showed that only exponential bounds are possible.

Height functions

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For instance, every point in $\mathbf{0} \neq \boldsymbol{x} \in \mathbb{Q}^n$ can be written as

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$$\boldsymbol{x} = \left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0} \right).$$

Define $d = \gcd(x_1, \ldots, x_n)$, then

$$H(\boldsymbol{x}) := \frac{1}{d} \max \{ |x_1|, \ldots, |x_n| \},$$

and so $H(a\boldsymbol{x}) = H(\boldsymbol{x})$ for every $0 \neq a \in \mathbb{Q}$. Hence H is *projectively defined*. We define $H(\mathbf{0}) = 0$.

Schmidt's height on subspaces

We can also talk about height of subspaces of K^n, as first introduced by **W. M. Schmidt** (1967). Let $V \subseteq K^n$ be an ℓ-dimensional subspace, and let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_\ell$ be a basis for V.

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This definition does not depend on the choice of the basis.
Duality: If $A = (\boldsymbol{a}_1 \dots \boldsymbol{a}_\ell)^t$ is an $\ell \times n$ matrix over K such that

$$V = \{\boldsymbol{x} \in K^n : A\boldsymbol{x} = \boldsymbol{0}\},$$

then

$$H(V) = H(\boldsymbol{a}_1 \wedge \cdots \wedge \boldsymbol{a}_\ell).$$

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Clearly,

$$H(\boldsymbol{x}) \leqslant h(\boldsymbol{x})$$

for all $\boldsymbol{x} \in K^n$.

Finiteness property

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Northcott's theorem: *If K is a number field or a function field over a finite coefficient field, then for every $B \in \mathbb{R}_{>0}$ the sets*

$$\{[\boldsymbol{x}] \in \mathbb{P}(K^n) : H(\boldsymbol{x}) \leqslant B\}, \ \{\boldsymbol{x} \in K^n : h(\boldsymbol{x}) \leqslant B\}$$

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More generally, height measures *arithmetic complexity* (by analogy with *degree* in algebraic geometry measuring *geometric complexity*), and so a point of relatively small height is *arithmetically simple*. This makes search bounds on height interesting even when Northcott's theorem fails.

Over more general fields

Analogues of Cassels' theorem over a global field K with respect to appropriately defined height functions have been obtained:

- In 1975 by **S. Raghavan** when K is a number field
- In 1987 by **A. Prestel** when K is a rational function field
- In 1997 by **A. Pfister** when K is an algebraic function field
- In 2008 by **L. F.** when $K = \overline{\mathbb{Q}}$

The exponent on height of F in the upper bounds is again $\frac{n-1}{2}$, except the $\overline{\mathbb{Q}}$ case where it is $1/2$.

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There have been many extensions and generalizations of Cassels' theorem over the years. An overview of these is presented in my survey paper:

Heights and quadratic forms: on Cassels' theorem and its generalizations, in "Diophantine methods, lattices, and arithmetic theory of quadratic forms" (W. K. Chan, L. Fukshansky, R. Schulze-Pillot, and J. D. Vaaler, eds.), Contemporary Mathematics, AMS vol. 587 (2013), pg. 77–94

Questions of distribution

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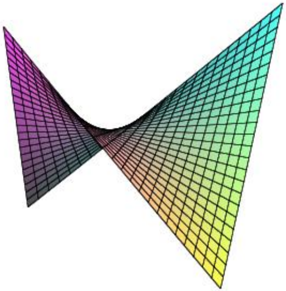

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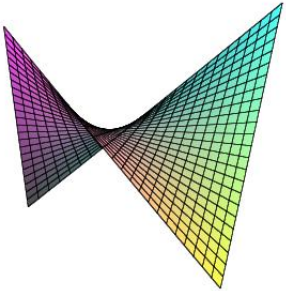

Avoiding varieties: homogeneous case

Theorem 2 (Gaudron, Remond - 2017)

Let K be a number field, F be a nonzero quadratic form in n variables over K, V be an m-dimensional subspace of K^n, and $w \geqslant 1$ be the dimension of a maximal totally isotropic subspace of the quadratic space (V, F). Let $\mathcal{Z}$ be a homogeneous algebraic set such that F has a nontrivial zero in $V \setminus \mathcal{Z}$. Then there exists a nontrivial zero $\boldsymbol{x} \in V \setminus \mathcal{Z}$ of F such that

$$H(\boldsymbol{x}) \ll H(F)^{\frac{m-w+1}{2}} H(V)^2,$$

where implied constant depends on K, m and M = degree of $\mathcal{Z}$.

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where implied constant depends on K, m and M = degree of $\mathcal{Z}$.

The original result like this was obtained by **Chan, F., Henshaw (2014)**, but with slightly weaker exponents. We also have analogues of Theorems 2 over global function fields and $\overline{\mathbb{Q}}$ with slightly weaker bounds.

Avoiding varieties: inhomogeneous case over a field

Theorem 3 (Chan, F. - 2019)

Let F be a nonzero quadratic form in n variables over a number field K and let $V \subseteq K^n$ be an m-dimensional subspace, $m \leqslant n$. Let $w \geqslant 0$ be the dimension of a maximal totally isotropic subspace of the quadratic space (V, F). Let $0 \neq t \in F(V)$ and $\mathcal{Z}$ be an algebraic set which does not contain the zero set of the polynomial

$$F_t(X_1, \dots, X_n) := F(X_1, \dots, X_n) - t.$$

Then there exists a point $\boldsymbol{z} \in V \setminus \mathcal{Z}$ such that $F(\boldsymbol{z}) = t$ and

$$h(\boldsymbol{z}) \ll H(F_t)^{\frac{m-w+2}{2}} H(V)^2.$$

The implied constant in the inequality depends only on K, m, n and $M =$ degree of $\mathcal{Z}$.

Avoiding subspaces: inhomogeneous case over $\mathbb{Z}$

Theorem 4 (Chan, F. - 2019)

Let $F(X_1, \ldots, X_n) \in \mathbb{Z}[X_1, \ldots, X_n]$ be an integral quadratic form. Let $V \subseteq \mathbb{Q}^n$ be an m-dimensional subspace, $3 \leqslant m \leqslant n$, such that the quadratic space (V, F) is nonsingular and isotropic with $w \geqslant 1$ be the dimension of a maximal totally isotropic subspace. Let $W_1, \ldots, W_k$ be distinct hyperplanes of V. Then for any $0 \neq t \in F(V \cap \mathbb{Z}^n)$, there exists $\mathbf{z} \in (V \cap \mathbb{Z}^n) \setminus \bigcup_{i=1}^k W_i$ such that $F(\mathbf{z}) = t$ and, if $w = 1$,

$$h(\mathbf{z}) \ll h(F_t)^{(1+\frac{2}{m-2})p(m)+m+2+\frac{m+4}{m-2}} H(V)^{(2+\frac{4}{m-2})p(m)+5+\frac{8}{m-2}},$$

whereas when $w \geqslant 2$,

$$h(\mathbf{z}) \ll h(F_t)^{2p(m)+\frac{m-w+5}{2}} H(V)^{2p(m)+3}.$$

The exponent $p(m) \approx 5m + c$ as in Dietmann's theorem. The implied constants depend on m, n, k.

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Skolem reduction allows to convert a question about solubility of a general system of Diophantine equations into a question about a system of linear and quadratic equations, and hence a search bound for solutions of a general quadratic system may be in conflict with Matiyasevich's negative answer to Hilbert's 10th problem.

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However, over $\overline{\mathbb{Q}}$, some such results turn out to be possible.

System of quadratic forms over $\overline{\mathbb{Q}}$

Theorem 5 (F. - 2015)

Let $k \geqslant 2$ be an integer, $F_1, \dots, F_k$ be quadratic forms in n variables over $\overline{\mathbb{Q}}$, and let $V \subseteq \overline{\mathbb{Q}}^n$ be an ℓ-dimensional subspace, $n \geqslant \ell \geqslant \frac{k(k+1)}{2} + 1$. There exists $\mathbf{0} \neq \boldsymbol{z} \in V$ such that $F_m(\boldsymbol{z}) = 0$ for all $1 \leqslant m \leqslant k$ and

$$h(\boldsymbol{z}) \leqslant \left(3^{\frac{\ell^2}{2}} n^{\frac{3(\ell+1)}{2}} H(V)\right)^{20B_k^2/81} \left(\prod_{m=1}^{k-1} H(F_m)\right)^{B_k} H(F_k)^2,$$

where $B_2 = 9$ and

$$B_k = \frac{1}{4} \times 36^{2^{k-2}} \prod_{m=3}^{k} m^{2^{k-m+1}}$$

for all $k \geqslant 3$.

Inhomogeneous polynomials over $\overline{\mathbb{Q}}$

Theorem 6 (F. - 2015)

Let F and G be quadratic polynomials in $n \geqslant 4$ variables over $\overline{\mathbb{Q}}$, possibly inhomogeneous. Let m be an integer, $0 \leqslant m \leqslant n-4$, and $\mathcal{L}_1, \dots, \mathcal{L}_m$ be linear polynomials in n variables over $\overline{\mathbb{Q}}$, possibly inhomogeneous; the case $m = 0$ just means that there are no linear polynomials. Suppose that the system

$$F(\boldsymbol{x}) = G(\boldsymbol{x}) = \mathcal{L}_1(\boldsymbol{x}) = \cdots = \mathcal{L}_m(\boldsymbol{x}) = 0$$

has a nontrivial solution over $\overline{\mathbb{Q}}$. Then there exists a point $\mathbf{0} \neq \boldsymbol{y} \in \overline{\mathbb{Q}}^n$ such that $F(\boldsymbol{y}) = \mathcal{L}_1(\boldsymbol{y}) = \cdots = \mathcal{L}_m(\boldsymbol{y}) = 0$ and

$$h(\boldsymbol{y}) \leqslant 8(n+1)^{2m} 3^{2(n-m+1)(n-m)} H(F)^{\frac{1}{2}} \prod_{i=1}^{m} H(\mathcal{L}_i)^4.$$

Inhomogeneous polynomials over $\overline{\mathbb{Q}}$

Theorem 6, continuation

There also exists a point $\mathbf{0} \neq \boldsymbol{z} \in \overline{\mathbb{Q}}^n$ *such that*

$$F(\boldsymbol{z}) = G(\boldsymbol{z}) = \mathcal{L}_1(\boldsymbol{z}) = \cdots = \mathcal{L}_m(\boldsymbol{z}) = 0$$

and

$$h(\boldsymbol{z}) \leqslant \mathcal{M}(m,n) H(F)^{58} H(G)^3 \prod_{i=1}^{m} H(\mathcal{L}_i)^{180},$$

where

$$\mathcal{M}(m,n) = 18 \times 8^{38} (n+1)^{90m+8} (n+1-m)^{36} 3^{90(n-m+1)(n-m)}.$$

Over a fixed number field

Our method can also be used to obtain analogues of Theorems 5 and 6 with points in question having bounded degree over a fixed number field. By Northcott's property, this provides actual search bounds for zeros of systems of quadratic and linear equations as above. On the other hand, the bounds on height we can obtain this way are weaker.

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http://math.cmc.edu/lenny/research.html


Thank You!
СПАСИБО!