Lattices from group frames and vertex transitive graphs

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(joint work with Deanna Needell, Josiah Park and Jessie Xin)

Tight frames

A spanning set $\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n\} \subset \mathbb{R}^k$, $n \geqslant k$, is called a **tight frame** if there exists a real constant γ such that for every $\boldsymbol{x} \in \mathbb{R}^k$,

$$\|\boldsymbol{x}\|^2 = \gamma \sum_{j=1}^{n} (\boldsymbol{x}, \boldsymbol{f}_j)^2,$$

where (,) stands for the usual dot-product.

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Tight frame $\mathcal{F}$ is **rational** if there exists a real number α so that

$$\alpha(\boldsymbol{f}_i, \boldsymbol{f}_j) \in \mathbb{Q} \ \forall \ 1 \leqslant i, j \leqslant n.$$

Group frames

Let G be a finite subgroup of $\mathcal{O}_k(\mathbb{R})$, the k-dimensional real orthogonal group, then G acts on $\mathbb{R}^k$ by left matrix multiplication. This action is **irreducible** if it has no invariant subspaces except for $\{\mathbf{0}\}$ and $\mathbb{R}^k$.

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Let $\boldsymbol{f} \in \mathbb{R}^k$ be a nonzero vector, then

$$G\boldsymbol{f} = \{U\boldsymbol{f} : U \in G\}$$

is called a **group frame** (or G-**frame**). If G acts irreducibly on $\mathbb{R}^k$, $G\boldsymbol{f}$ is called an **irreducible group frame**. Irreducible group frames are always tight.

Tight frames and lattices

Let $\mathcal{F} = \{\boldsymbol{f}_1, \dots, \boldsymbol{f}_n\}$ be a tight (n, k)-frame and define

$$L(\mathcal{F}) := \operatorname{span}_{\mathbb{Z}} \mathcal{F} = \left\{ \sum_{i=1}^{n} a_i \boldsymbol{f}_i : a_1, \dots, a_n \in \mathbb{Z} \right\}.$$

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What kind of lattices can we get this way? What kind of lattices do we want to get?

Some properties of lattices

Minimal norm of a lattice L is

$$|L| = \min \{\|\boldsymbol{x}\| : \boldsymbol{x} \in L \backslash \{\mathbf{0}\}\},$$

where $\| \ \|$ is Euclidean norm. The set of **minimal vectors** of L is

$$S(L) = \{\boldsymbol{x} \in L : \|\boldsymbol{x}\| = |L|\}.$$

From here on, let us write $S(L) = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_n\}$, where n, the cardinality of the set of minimal vectors of L must be even, and can be smaller or larger than (twice) the rank of the lattice. For example

$$S(\mathbb{Z}^k) = \{\pm\boldsymbol{e}_1, \dots, \pm\boldsymbol{e}_k\},$$

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so $n = 2k$. A lattice L is **well-rounded** (WR) if

$$\operatorname{span}_{\mathbb{R}} L = \operatorname{span}_{\mathbb{R}} S(L).$$

Eutaxy and perfection

A lattice $L \subset \mathbb{R}^k$ is called **eutactic** if there exist positive real numbers $c_1, \ldots, c_n$ such that

$$\|\boldsymbol{v}\|^2 = \sum_{i=1}^{n} c_i (\boldsymbol{v}, \boldsymbol{x}_i)^2$$

for every vector $\boldsymbol{v} \in \mathbb{R}^k$. If $c_1 = \cdots = c_n$, we say that L is **strongly eutactic**.

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Both, eutactic and perfect lattices are necessarily well-rounded. Up to similarity, there are only finitely many eutactic or perfect lattices in a given dimension.

Packing density

The **packing density** of a lattice L of rank k is defined as

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Theorem 2 (G. Voronoi, 1908)

A lattice is extremal if and only if it is perfect and eutactic.

Strong eutaxy and group frames

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a lattice $L \subset \mathbb{R}^k$ is strongly eutactic if and only if its set of minimal vectors $S(L)$ forms a tight frame in $\mathbb{R}^k$.

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Theorem 3 (F., Needell, Park, Xin (2018))

Let G be a group of $k \times k$ real orthogonal matrices and $\boldsymbol{f} \in \mathbb{R}^k$ be a vector so that $\mathcal{F} = G\boldsymbol{f}$ is an irreducible rational group frame in $\mathbb{R}^k$. Then the lattice $L(\mathcal{F})$ is strongly eutactic.

Transitive graphs

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We define the **distance** between two vertices in a graph to be the number of edges in a shortest path connecting them. A connected graph Γ is called **distance transitive** if for any two pairs of vertices i, j and k, l at the same distance from each other there exists an automorphism $\tau \in G$ such that $\tau(i) = k$ and $\tau(j) = l$.

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Since $G \leqslant S_n$, we can identify it with its representation in the orthogonal group $\mathcal{O}_n(\mathbb{R})$ and its action extends to $\mathbb{R}^n$: $\forall\ \tau \in G$,

$$\tau \boldsymbol{e}_i = \boldsymbol{e}_{\tau(i)},$$

where $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ are standard basis vectors.

Graphs to frames

Γ = distance transitive graph on n vertices

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Hence, if $\boldsymbol{f} \in V_\lambda$, $\boldsymbol{f} \neq \mathbf{0}$, then $G\boldsymbol{f}$ is an irreducible group frame. Since there are many distance transitive graphs with rational eigenvalues, this becomes a valuable source for our lattice construction, leading to the following result.

Lattices from graphs

Theorem 4 (F., Needell, Park, Xin (2018))

Let Γ be a vertex transitive graph on n vertices and G its automorphism group. Let A be the adjacency matrix of Γ, λ a rational eigenvalue of multiplicity m and V_λ the corresponding m-dimensional eigenspace. Let P_λ be a rational orthogonal projection matrix of $\mathbb{R}^n$ onto V_λ. Then

$$L_{\Gamma,\lambda} := P_\lambda \mathbb{Z}^n$$

is a lattice of full rank in V_λ, and its automorphism group contains a subgroup isomorphic to a factor group of G. If Γ is distance transitive, $L_{\Gamma,\lambda}$ is strongly eutactic.

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We will refer to lattices obtained in this way as **lattices generated by graphs**. All vertex transitive examples known to us also generate strongly eutactic lattices, but we do not have a proof that this is true in general.

Hadwiger's Principal Theorem

We compare our result to a classical theorem of H. Hadwiger.

Theorem 5

A set $S = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ of cardinality n in k-dimensional space V, $n > k$, is eutactic, i.e. there exist positive real numbers $c_1, \dots, c_n$ such that

$$\|\mathbf{v}\|^2 = \sum_{i=1}^{n} c_i (\mathbf{v}, \mathbf{x}_i)^2$$

for every vector $\mathbf{v} \in \mathbb{R}^k$ if and only if it is an orthogonal projection onto V of an orthonormal basis in an n-dimensional space containing V.

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Our graph construction considers precisely such a projection, namely the set of vectors $\{P_\lambda \boldsymbol{e}_i\}_{i=1}^n$ where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ is the standard basis in $\mathbb{R}^n$. This set is therefore eutactic by Hadwiger.

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Basic properties of this construction

Theorem 6 (F., Needell, Park, Xin (2018))

- *The completely disconnected graph* 0_n *on* n *vertices generates the integer lattice* $\mathbb{Z}^n$.
- *The complete graph* K_n *generates (a lattice similar to) the root lattice*

$$A_{n-1} := \left\{ \boldsymbol{x} \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i = 0 \right\}.$$

- *If* Γ *is a disjoint union of* k *copies of a vertex transitive graph* Δ *with a rational eigenvalue* λ*, then* λ *is also an eigenvalue of* Γ *and*

$$L_{\Gamma,\lambda} = L_{\Delta,\lambda} \perp \cdots \perp L_{\Delta,\lambda}.$$

More properties

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Theorem 7 (F., Needell, Park, Xin (2018))

Let Γ be a vertex transitive graph on n vertices of degree k and Γ' its complement. Then for each rational eigenvalue $\lambda \neq k$ of Γ there is a rational eigenvalue $\lambda' = -\lambda - 1$ of Γ' of the same multiplicity and the lattices

$$L_{\Gamma,\lambda} = L_{\Gamma',\lambda'}.$$

Product graphs

There are three standard commutative product operations on graphs:

- **Cartesian product:** $\Delta_1 \square \Delta_2$ is the graph whose vertices are pairs (u, v), where u is a vertex of Δ_1 and v is a vertex of Δ_2, and (u_1, v_1) and (u_2, v_2) are connected by an edge if and only if either $u_1 = u_2$ and v_1, v_2 are connected by an edge in Δ_2, or $v_1 = v_2$ and u_1, u_2 are connected by an edge in Δ_1.

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- **Direct product:** $\Delta_1 \times \Delta_2$ is the graph whose vertices are pairs (u, v), where u is a vertex of Δ_1 and v is a vertex of Δ_2, and two vertices (u_1, v_1) and (u_2, v_2) are connected by an edge if and only if both pairs u_1, u_2 and v_1, v_2 are connected by an edge in Δ_1, Δ_2, respectively.

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- **Strong product:** $\Delta_1 \boxtimes \Delta_2$, is the graph whose vertices are pairs (u, v), where u is a vertex of Δ_1 and v is a vertex of Δ_2, and two vertices (u_1, v_1) and (u_2, v_2) are connected by an edge if and only if u_1, u_2 and v_1, v_2 are either equal or connected by an edge in Δ_1, Δ_2, respectively.

Eigenvalues of product graphs

Products of vertex transitive graphs Δ_1 and Δ_2 are vertex transitive. Further, suppose λ, μ are eigenvalues of Δ_1, Δ_2, respectively. Then:

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- **Cartesian product:**

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  is an eigenvalue of $\Delta_1 \square \Delta_2$.

- **Direct product:**

$$f(\lambda, \mu) = \lambda\mu$$

  is an eigenvalue of $\Delta_1 \times \Delta_2$.

- **Strong product:**

$$f(\lambda, \mu) = (\lambda + 1)(\mu + 1) - 1$$

  is an eigenvalue of $\Delta_1 \boxtimes \Delta_2$.

Lattices from product graphs

Theorem 8 (F., Needell, Park, Xin (2018))

Let Δ_1, Δ_2 *be vertex transitive graphs on* m_1, m_2 *vertices, respectively, and let* Γ *be a product graph* $\Gamma = \Delta_1 * \Delta_2$ *on* $m_1 m_2$ *vertices, where* $*$ *stands for* $\square$, $\times$, *or* $\boxtimes$. *Let* ν *be an eigenvalue of* Γ *and* (λ_i, μ_i) *for* $1 \leqslant i \leqslant k$ *pairs of eigenvalues of* Δ_1, Δ_2 *respectively so that* $\nu = f(\lambda_i, \mu_i)$ *for all* $1 \leqslant i \leqslant k$ *for the appropriate* f. *Let* L_{Δ_1,λ_i} *and* L_{Δ_2,μ_i} *for each* $1 \leqslant i \leqslant k$ *be the corresponding lattices. Then* $L_{\Gamma,\nu}$ *is the orthogonal projection of* $\mathbb{Z}^{m_1 m_2}$ *onto the space spanned by*

$$\left(L_{\Delta_1,\lambda_1} \otimes_{\mathbb{Z}} L_{\Delta_2,\mu_1}\right) \perp \cdots \perp \left(L_{\Delta_1,\lambda_k} \otimes_{\mathbb{Z}} L_{\Delta_2,\mu_k}\right),$$

where $\perp$ *is the orthogonal direct sum. In particular, if* $k = 1$ *then*

$$L_{\Gamma,\nu} = L_{\Delta_1,\lambda_1} \otimes_{\mathbb{Z}} L_{\Delta_2,\mu_1},$$

up to similarity.

And now examples!

Graph Γ	**# of vertices**	**Eig. λ**	**Mult. of λ**	**Lattice $L_{\Gamma,\lambda}$**
Cube Q_3	(8)	1	(3)	A_3^*, dual of A_3
Hamming $H(2,3)$	(9)	1	(4)	$A_2 \otimes_{\mathbb{Z}} A_2$
Petersen graph	(10)	-2	(4)	A_4^*, dual of A_4
Petersen graph	(10)	1	(5)	Coxeter lattice A_5^2
Petersen line graph	(15)	-1	(5)	A_4^*, dual of A_4
Petersen line graph	(15)	-2	(5)	Coxeter lattice A_5^3
Clebsch graph	(16)	-3	(5)	D_5^*, dual of D_5
Shrikhande graph	(16)	2	(6)	D_6^+
Schläfli graph	(27)	4	(6)	E_6^*, dual of E_6
Gosset graph	(56)	9	(7)	E_7^*, dual of E_7

Examples of strongly eutactic lattices from vertex transitive graphs

(**Dual:** $L^* := \{\boldsymbol{x} \in \mathbb{R}^n : (\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{Z} \text{ for all } \boldsymbol{y} \in L\}$)

Lattices E_8, E_7, E_6

For each $n \geqslant 2$,

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Then $E_8 := D_8^+$, and

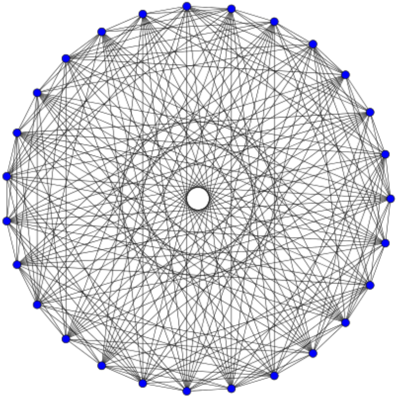
$$E_7 := \{\boldsymbol{x} \in E_8 : x_7 = x_8\}, \ E_6 := \{\boldsymbol{x} \in E_8 : x_6 = x_7 = x_8\}.$$

Their duals are

$$E_7^* = \{\boldsymbol{x} \in \operatorname{span}_{\mathbb{R}} E_7 : (\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{Z} \ \forall \ \boldsymbol{y} \in E_7\},$$

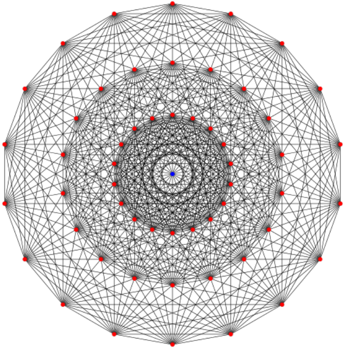
$$E_6^* = \{\boldsymbol{x} \in \operatorname{span}_{\mathbb{R}} E_6 : (\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{Z} \ \forall \ \boldsymbol{y} \in E_6\}.$$

Schläfli graph



Γ is the complement of the intersection graph of the 27 lines on a cubic surface. It is a strongly regular graph on 27 vertices with parameters $k = 16$, $\ell = 10$, $m = 8$ and has eigenvalue 4 of multiplicity 6: $L_{\Gamma,4} = E_6^*$.

Gosset graph



Γ has 56 vertices corresponding to the set of edges of two copies of the complete graph K_8. Two vertices from the same copy of K_8 are connected by an edge if they correspond to disjoint edges of K_8; two vertices from different copies of K_8 are are connected by an edge if they correspond to edges that meet in a vertex. Γ has eigenvalue 9 of multiplicity 7: $L_{\Gamma,9} = E_7^*$.

Contact polytope

Contact polytope of a lattice L is

$$C(L) = \text{Conv}\{S(L)\}.$$

Its vertices are points on the sphere centered at the origin in the sphere packing associated to L at which neighboring spheres touch it. Hence the number of vertices of $C(L)$ is the kissing number of L.

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The **skeleton graph** of the polytope $C(L)$, $\text{skel}(C(L))$ is the graph consisting of vertices and edges of $C(L)$.

A curious duality

Let $L = E_6^*$. The contact polytope of E_6^* has 54 vertices, split into 27 $\pm$ pairs: it is a diplo-Schläfli polytope. The Schläfli polytope, has 27 vertices corresponding to the 27 lines on a cubic surface. Its skeleton is the Schläfli graph Γ, which has an eigenvalue 4 of multiplicity 6, and $L_{\Gamma,4} = E_6^*$.

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Let $L = E_7^*$, its contact polytope is the Gosset polytope (also called Hess polytope), which has 56 vertices. Its skeleton is the Gosset graph Γ, which has an eigenvalue 9 of multiplicity 7, and $L_{\Gamma,9} = E_7^*$.

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Let $L = E_7^*$, its contact polytope is the Gosset polytope (also called Hess polytope), which has 56 vertices. Its skeleton is the Gosset graph Γ, which has an eigenvalue 9 of multiplicity 7, and $L_{\Gamma,9} = E_7^*$.

Let $L = A_3^*$. Its contact polytope is a cube (which is a diplo-simplex), whose skeleton graph Q_3 has eigenvalue 1 (and -1) of multiplicity 3, and $L_{Q_3,1} = A_3^*$.

Reference

L. Fukshansky, D. Needell, J. Park, Y. Xin. *Lattices from tight frames and vertex transitive graphs*, Electronic Journal of Combinatorics, vol. 26 no. 3 (2019), P3.49, 30 pp.

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