Height bounds for zeros of quadratic forms

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Diophantine Approximation and Related Topics
Aahrus, Denmark, July 13-17, 2015

Cassels' Theorem

Let $N \geq 2$, and let

$$F(\boldsymbol{X}, \boldsymbol{Y}) = \sum_{i=1}^{N} \sum_{j=1}^{N} f_{ij} X_i Y_j$$

be a symmetric bilinear form with integer coefficients $f_{ij} = f_{ji}$. Write $F(\boldsymbol{X}) := F(\boldsymbol{X}, \boldsymbol{X})$ for the associated quadratic form.

A famous result of **J. W. S. Cassels** (1955) states that if F is isotropic over $\mathbb{Q}$, then it has a nontrivial integral zero $\boldsymbol{x}$ such that

$$|\boldsymbol{x}| \ll_N |F|^{\frac{N-1}{2}}, \tag{1}$$

where

$$|\boldsymbol{x}| := \max_{1 \leq i \leq N} |x_i|, \quad |F| := \max_{1 \leq i,j \leq N} |f_{ij}|,$$

and the constant in the upper bound is explicit. The exponent $\frac{N-1}{2}$ in the upper bound is best possible, as shown by an example due to **M. Knesser**.

Over more general fields

Analogues of Cassels' theorem over a global field K with respect to appropriately defined height functions have been obtained:

- In 1975 by **S. Raghavan** when K is a number field
- In 1987 by **A. Prestel** when K is a rational function field
- In 1997 by **A. Pfister** when K is an algebraic function field

The exponent on height of F in the upper bounds is again $\frac{N-1}{2}$.

There have been many other extensions and generalizations of Cassels' theorem. An overview of this area can be found in my recent survey paper:

Heights and quadratic forms: on Cassels' theorem and its generalizations, in "Diophantine methods, lattices, and arithmetic theory of quadratic forms" (W. K. Chan, L. Fukshansky, R. Schulze-Pillot, and J. D. Vaaler, eds.), Contemporary Mathematics, AMS vol. 587 (2013), pg. 77–94

Here we discuss an effective theory of quadratic forms with respect to height that followed Cassels'-type results.

Fields and absolute values

- $\mathfrak{K} = \mathbb{Q}$ or $\mathbb{F}_q(t)$ for some prime power q
- K = a finite extension of $\mathfrak{K}$
- $d = [K : \mathfrak{K}]$ = global degree of $K/\mathfrak{K}$
- $M(K)$ = set of all places of K
- $\forall\ v \in M(K)$, K_v = completion of K at v, $d_v = [K_v : \mathfrak{K}_v]$
- $\forall\ u \in M(\mathfrak{K})$, $\sum_{v \in M(K), v|u} d_v = d$
- We choose absolute values $\forall\ v \in M(K)$ so that

$$\prod_{v \in M(K)} |a|_v^{d_v} = 1,$$

for all $0 \neq a \in K$.

Projective height

Let $N \geq 1$. For each $v \in M(K)$, define the local sup-norm

$$|\boldsymbol{x}|_v = \max_{1 \leq i \leq N} |x_i|_v$$

on K_v^N, and for each $v \mid \infty$ also define the local L_2-norm

$$\|\boldsymbol{x}\|_v = \left(\sum_{i=1}^{N} |x_i|_v^2 \right)^{1/2}.$$

Define **projective height function** H on K^N by

$$H(\boldsymbol{x}) = \prod_{v \in M(K)} |\boldsymbol{x}|_v^{d_v/d}$$

for each $\boldsymbol{x} \in K^N$.

For a polynomial F with coefficients in K, $H(F)$ is the height of its coefficient vector.

Schmidt height on subspaces

Let $V \subseteq K^N$ be an L-dimensional subspace, $1 \leq L \leq N$, and let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_L$ be a basis for V. Let

$$\boldsymbol{y} := \boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_L \in K^{\binom{N}{L}}$$

under the standard embedding. Define

$$\mathcal{H}(V) := \prod_{v \nmid \infty} |\boldsymbol{y}|_v^{d_v/d} \times \prod_{v \mid \infty} \|\boldsymbol{y}\|_v^{d_v/d}.$$

This definition is independent of the choice of the basis by the product formula. It was first given by **W. M. Schmidt (1967)**.

Remark 1

The normalizing exponent $1/d$ in the definition of H and $\mathcal{H}$ ensures that our heights are **absolute**, i.e. do not depend on the field of definition (in other words, defined over $\overline{\mathfrak{K}}$).

Quadratic spaces

Let $V \subseteq K^N$ be an L-dimensional subspace, F a quadratic form in $N \geq 2$ variables over K isotropic on V. We will call the pair (V, F) an L-dimensional quadratic space in N-variables over K. The **radical** of (V, F) is

$$V^{\perp} := \{\boldsymbol{x} \in V : F(\boldsymbol{x}, \boldsymbol{y}) = 0 \ \forall \ \boldsymbol{y} \in V\},$$

i.e. the space of all singular points in (V, F). Then

$$V = V^{\perp} \perp W,$$

where (W, F) is nonsingular (regular). We write $\lambda = \dim_K V^{\perp}$ and $r = L - \lambda$, the **rank** of F on V.
A subspace $U \subseteq V$ is called **totally isotropic** if $F(U) = 0$. The common dimension ω of all maximal totally isotropic subspaces of W is called the **Witt index** of (V, F), and hence maximal totally isotropic subspaces of V have dimension $M := \lambda + \omega$.

Isotropic subspaces of bounded height

The following theorem was proved by **Schlickewei (1985)** over $\mathbb{Q}$, by **Vaaler (1987)** over number fields, and by **Chan, F., Henshaw (2014)** over global function fields.

Theorem 1

Let (V, F) be an L-dimensional quadratic space in N variables over K and let $M \geq 1$ be the dimension of its maximal totally isotropic subspaces. Then there exists a maximal totally isotropic subspace U of V such that

$$\mathcal{H}(U) \ll_{K,L,M} H(F)^{\frac{L-M}{2}} \mathcal{H}(V).$$

Remark 2

Analogous result over $\overline{\mathbb{Q}}$, but with weaker exponents, was obtained by **F. (2008)**.

Collections of isotropic subspaces

More generally, there exist collections of totally isotropic subspaces generating the full quadratic space. The following theorem was proved by **Schlickewei and Schmidt (1987)** over $\mathbb{Q}$ and by **Vaaler (1989)** over number fields.

Theorem 2

Let (V, F) be an L-dimensional quadratic space in N variables over K and let $M \geq 1$ be the dimension of its maximal totally isotropic subspaces. There exists a collection of $L - M + 1$ maximal totally isotropic subspaces $U_0, \dots, U_{L-M} \subseteq V$ such that $V = \operatorname{span}_K \{U_0, \dots, U_{L-M}\}$, and for each $0 \leq i \leq L - M$,

$$H(U_0)H(U_i) \ll_{K,L,M} H(F)^{L-M}\mathcal{H}(V)^2.$$

Infinite family

Here is an extension of the Schlickewei-Schmidt-Vaaler theorem, although with weaker bounds, holding over **any global field or** $\overline{\mathbb{Q}}$.

Theorem 3 (Chan, F., Henshaw (2014))

Let (V, F) *be an* L*-dimensional quadratic space in* N *variables over* K *of Witt index* $\omega \geq 1$ *and dimension of the radical* $\lambda \geq 0$*. There exists an infinite family of collections of maximal totally isotropic subspaces* $\{U_{n1}, \dots, U_{nJ}\}_{n=1}^{\infty} \subseteq V$*, for an appropriately defined* J*, such that for each* $n \geq 1$, $\operatorname{span}_K \{U_{n1}, \dots, U_{nJ}\} = V$*, and for each* $1 \leq j \leq J$*,*

$$H(U_{nj}) \ll H(F)^{\mathrm{p}(L,\omega)} H(V)^{\mathrm{q}(\omega)},$$

where the constant in the upper bound depends on $K, N, L, \omega, \lambda, n$*, and* $\mathrm{p}(L,\omega)$, $\mathrm{q}(\omega)$ *are polynomials:* $\mathrm{p}(L,\omega)$ *is linear in* L*, quartic in* ω*, and* $\mathrm{q}(\omega)$ *is cubic in* ω*. The constant depending on* n *is* n^2 *if* K *is a number field,* e^{2n} *if* K *is a function field, and 1 if* $K = \overline{\mathbb{Q}}$*.*

Witt decomposition

Let K be a number field, global function field, or $\overline{\mathbb{Q}}$, and let (V, F) be an L-dimensional quadratic space in N variables over K. We use the same notation as above:

$$\lambda = \dim_K V^{\perp},\ r = L - \lambda,\ \omega = \text{ Witt index of } (V, F).$$

A 2-dimensional subspace $\operatorname{span}_K \{\boldsymbol{x}, \boldsymbol{y}\}$ of (V, F) is called a **hyperbolic plane** if $F(\boldsymbol{x}) = F(\boldsymbol{y}) = 0,\ F(\boldsymbol{x}, \boldsymbol{y}) \neq 0$.
A subspace W of (V, F) is called **anisotropic** if $F(\boldsymbol{x}) \neq 0$ for any $\boldsymbol{0} \neq \boldsymbol{x} \in W$.

Theorem 4 (Witt decomposition theorem)

There exists an orthogonal decomposition of (V, F)

$$V = V^{\perp} \perp \mathbb{H}_1 \perp \ldots \perp \mathbb{H}_{\omega} \perp W,$$

where $\mathbb{H}_1, \ldots, \mathbb{H}_{\omega}$ *are hyperbolic planes and* W *is an anisotropic component.*

Effective Witt decomposition

It has been shown by **Vaaler (1989)** that over global fields

$$\mathcal{H}(V^{\perp}) \ll_{K,L,r} H(F)^{\frac{r}{2}}\, \mathcal{H}(V).$$

Witt decomposition is unique up to isometry, but isometry can change height of nonsingular components arbitrarily.

Theorem 5 (F. (2007/2014))

There exists a Witt decomposition of the quadratic space (V, F) over a global field K such that

$$\max\{\mathcal{H}(\mathbb{H}_i), \mathcal{H}(W)\} \ll_{K,N,L,\omega} \left\{ H(F)^{\frac{2\omega+r}{4}} \mathcal{H}(V) \right\}^{\frac{(\omega+1)(\omega+2)}{2}}.$$

Remark 3

Analogous result over $\overline{\mathbb{Q}}$ (with weaker bounds) has been obtained by **F. (2008)**.

Inhomogeneous quadratic polynomials

An analogue of Cassels' theorem for rational zeros of an inhomogeneous rational quadratic polynomial has been proved by **Masser (1998)**.

Theorem 6 (Masser (1998))

Let $F(\boldsymbol{X})$ be an inhomogeneous quadratic polynomial in $N \geq 2$ variables with rational coefficients that has a nontrivial rational zero. Then it has such a zero $\boldsymbol{x}$ with

$$H(\boldsymbol{x}) \ll_N H(F)^{(N+1)/2}.$$

The exponent in the upper bound is sharp.

The idea of the proof is to introduce an additional variable X_0 to homogenize F, and then apply Cassels' theorem while ensuring that $X_0 \neq 0$.

Missing subspaces

Masser's theorem can then be viewed as a result on existence of small-height zeros of quadratic forms outside of subspace defined by a particular linear form.

An extension of Masser's theorem over number fields and its generalization to zeros of a quadratic form "missing" any finite collection of subspaces was obtained by **F. (2004)**, and then with improved bounds by **Dietmann (2009)**.

A more general result of this kind was obtained recently. We state it over global fields, but there is also a version over $\overline{\mathbb{Q}}$ with weaker bounds. This result can be loosely interpreted as a distribution statement on small-height zeros of a quadratic form: they cannot be easily "cut out" by finite collections of polynomial maps.

Missing varieties

Theorem 7 (Chan, F., Henshaw (2014))

Let (V, F) be an isotropic quadratic space of dimension L in N variables over a global field K and let $M = \omega + \lambda \geq 1$ be the dimension of its maximal totally isotropic subspace. Let $\mathcal{Z}_K$ be a union of a finite collection of projective varieties defined over K such that F has a nontrivial zero in $V \not\subseteq \mathcal{Z}_K$, and let D be sum of degrees of these varieties. Then there exist M linearly independent zeros $\boldsymbol{x}_1, \dots, \boldsymbol{x}_M$ of F in $V \setminus \mathcal{Z}_K$ such that for each $1 \leq n \leq M$,

$$H(\boldsymbol{x}_n) \ll_{K,L,M,D} H(F)^{\frac{9L+11}{2}} \mathcal{H}(V)^{9L+12}.$$

Moreover, there exists a chain of totally isotropic subspaces $W_1 \subset \dots \subset W_M$ of (V, F) with $\dim_K W_n = n$ and $W_n \not\subseteq \mathcal{Z}_K$ for each $1 \leq n \leq M$ such that

$$\mathcal{H}(W_n) \ll_{K,L,M,D} H(F)^{15(L+1)-M} \mathcal{H}(V)^{27L+37}.$$

Some further work

Some analogous results were also obtained for symplectic spaces over number fields, function fields, and their algebraic closures (**F., 2009**), and for hermitian forms over positive definite quaternion algebras over totally real number fields (**Chan, F. (2010); F., Henshaw (2013)**.

Improved bounds for small-height isotropic subspaces (Theorem 1), Witt decomposition (Theorem 5), and zeros missing subspaces (Theorem 7) were recently obtained by **Gaudron & Remond (2015)** over number fields and (partially) over $\overline{\mathbb{Q}}$ (but not over function fields).

Obtaining height bounds for systems of quadratic equations over a fixed number field currently appears to be out of reach, however some results are possible over $\overline{\mathbb{Q}}$. We discuss this next.

System of quadratic forms over $\overline{\mathbb{Q}}$

Theorem 8 (F., 2015)

Let $k \geq 1$ be an integer, $F_1, \dots, F_k$ be quadratic forms in N variables over $\overline{\mathbb{Q}}$, and let $V \subseteq \overline{\mathbb{Q}}^N$ be an L-dimensional subspace, $N \geq L \geq \frac{k(k+1)}{2} + 1$. There exists $\mathbf{0} \neq \boldsymbol{w} \in V$ such that $F_m(\boldsymbol{w}) = 0$ for all $1 \leq m \leq k$ and

$$H(\boldsymbol{w}) \ll_L \mathcal{H}(V)^{4/L} H(F)^{1/2},$$

if $k = 1$; if $k \geq 2$,

$$H(\boldsymbol{w}) \ll_{L,N,k} \mathcal{H}(V)^{20B_k^2/81} H(F_k)^2 \prod_{n=1}^{k-1} H(F_n)^{B_k},$$

where $B_2 = 9$ and for all $k \geq 3$, $B_k = \frac{1}{4} \times 36^{2^{k-2}} \prod_{m=3}^{k} m^{2^{k-m+1}}$.

Inhomogeneous polynomials over $\overline{\mathbb{Q}}$

Theorem 9 (F., 2015)

Let F and G be quadratic polynomials in $N \geq 4$ variables over $\overline{\mathbb{Q}}$, possibly inhomogeneous. Let m be an integer, $0 \leq m \leq N-4$, and $\mathcal{L}_1, \ldots, \mathcal{L}_m$ be linear polynomials in N variables over $\overline{\mathbb{Q}}$, possibly inhomogeneous; the case $m = 0$ just means that there are no linear polynomials. Suppose that the system

$$F(\boldsymbol{x}) = G(\boldsymbol{x}) = \mathcal{L}_1(\boldsymbol{x}) = \cdots = \mathcal{L}_m(\boldsymbol{x}) = 0$$

has a nontrivial solution over $\overline{\mathbb{Q}}$. Then there exists a point $\mathbf{0} \neq \boldsymbol{y} \in \overline{\mathbb{Q}}^N$ such that $F(\boldsymbol{y}) = \mathcal{L}_1(\boldsymbol{y}) = \cdots = \mathcal{L}_m(\boldsymbol{y}) = 0$ and

$$H(\boldsymbol{y}) \ll_{N,m} H(F)^{\frac{1}{2}} \prod_{i=1}^{m} H(\mathcal{L}_i)^4.$$

Inhomogeneous polynomials over $\overline{\mathbb{Q}}$

Theorem 9, continuation

There also exists a point $\mathbf{0} \neq \boldsymbol{w} \in \overline{\mathbb{Q}}^N$ *such that*

$$F(\boldsymbol{w}) = G(\boldsymbol{w}) = \mathcal{L}_1(\boldsymbol{w}) = \cdots = \mathcal{L}_m(\boldsymbol{w}) = 0$$

and

$$H(\boldsymbol{w}) \ll_{N,m} H(F)^{58} H(G)^3 \prod_{i=1}^{m} H(\mathcal{L}_i)^{180}.$$

Over a fixed number field

Our method can also be used to obtain analogues of Theorems 8 and 9 with points in question having bounded degree over a fixed number field. By Northcott's property, sets of projective points over $\overline{\mathbb{Q}}$ of bounded height and degree are finite, hence this provides actual search bounds for zeros of systems of quadratic and linear equations as above. On the other hand, the bounds on height we can obtain this way are weaker.

Thank you!