Heights and effective theory of quadratic forms over global fields

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Cassels' Theorem

Let $N \geq 2$, and let

$$F(\mathbf{X}, \mathbf{Y}) = \sum_{i=1}^{N} \sum_{j=1}^{N} f_{ij} X_i Y_j$$

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A famous result of **J. W. S. Cassels** (1955) states that if F is isotropic over $\mathbb{Q}$, then it has a nontrivial integral zero $\mathbf{x}$ such that

$$|\mathbf{x}| \ll_N |F|^{\frac{N-1}{2}}, \tag{1}$$

where

$$|\mathbf{x}| := \max_{1 \leq i \leq N} |x_i|, \quad |F| := \max_{1 \leq i,j \leq N} |f_{ij}|,$$

and the constant in the upper bound is explicit. The exponent $\frac{N-1}{2}$ in the upper bound is best possible, as shown by an example due to **M. Knesser**.

Over more general fields

Analogues of Cassels' theorem over a global field K with respect to appropriately defined height functions have been obtained:

- In 1975 by **S. Raghavan** when K is a number field
- In 1987 by **A. Prestel** when K is a rational function field
- In 1997 by **A. Pfister** when K is an algebraic function field

The exponent on height of F in the upper bounds is again $\frac{N-1}{2}$. In addition, Raghavan produced an analogous result for zeros of **hermitian forms** over CM number fields, where the exponent in the upper bound is $\frac{2N-1}{2}$.

Further generalizations

Additional extensions of Cassels' theorem have been obtained by **Birch & Davenport (1958), Chalk (1980), Davenport (1957 & 1971), Schulze-Pillot (1983), Watson (1956)**, and others. In particular, results of Chalk, Davenport (1971) and Schulze-Pillot established existence theorems for collections of linearly independent isotropic vectors of bounded height.

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Remark 1

Recently **Browning & Dietmann (2008)** produced a better bound than Cassels' for *generic* quadratic forms. There are also several analogous results in the literature on small-height zeros of cubic forms, most recently by **Browning, Dietmann, & Elliott (2012)**. The case of linear forms is well-covered by **Siegel's lemma** and its numerous generalizations. Little is known in general about forms of degree > 3.

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In this talk, we discuss a more general effective theory of quadratic forms that followed Cassels'-type results.

Fields and absolute values

- $\mathfrak{K} = \mathbb{Q}$ or $\mathbb{F}_q(t)$ for some prime power q
- K = a finite extension of $\mathfrak{K}$
- $d = [K : \mathfrak{K}]$ = global degree of $K/\mathfrak{K}$
- $M(K)$ = set of all places of K
- $\forall\ v \in M(K)$, K_v = completion of K at v, $d_v = [K_v : \mathfrak{K}_v]$
- $\forall\ u \in M(\mathfrak{K})$, $\sum_{v \in M(K), v|u} d_v = d$
- We choose absolute values $\forall\ v \in M(K)$ so that

$$\prod_{v \in M(K)} |a|_v^{d_v} = 1,$$

for all $0 \neq a \in K$.

Projective height

Let $N \geq 1$. For each $v \in M(K)$, define the local sup-norm

$$|\mathbf{x}|_v = \max_{1 \leq i \leq N} |x_i|_v$$

on K_v^N, and for each $v \mid \infty$ also define the local L_2-norm

$$\|\mathbf{x}\|_v = \left(\sum_{i=1}^{N} |x_i|_v^2 \right)^{1/2}.$$

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For a polynomial F with coefficients in K, $H(F)$ is the height of its coefficient vector.

Schmidt height on subspaces

Let $V \subseteq K^N$ be an L-dimensional subspace, $1 \leq L \leq N$, and let $\mathbf{x}_1, \ldots, \mathbf{x}_L$ be a basis for V. Let

$$\mathbf{y} := \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_L \in K^{\binom{N}{L}}$$

under the standard embedding. Define

$$\mathcal{H}(V) := \prod_{v \nmid \infty} |\mathbf{y}|_v^{d_v/d} \times \prod_{v \mid \infty} \|\mathbf{y}\|_v^{d_v/d}.$$

This definition is independent of the choice of the basis by the product formula. It was first given by **W. M. Schmidt (1967)**.

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Remark 2

The normalizing exponent $1/d$ in the definition of H and $\mathcal{H}$ ensures that our heights are **absolute**, i.e. do not depend on the field of definition (in other words, defined over $\overline{\mathfrak{K}}$).

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$$V^{\perp} := \{\mathbf{x} \in V : F(\mathbf{x}, \mathbf{y}) = 0 \ \forall \ \mathbf{y} \in V\},$$

i.e. the space of all singular points in (V, F). Then

$$V = V^{\perp} \perp W,$$

where (W, F) is nonsingular (regular). We write $\lambda = \dim_K V^{\perp}$ and $r = L - \lambda$, the **rank** of F on V.

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A subspace $U \subseteq V$ is called **totally isotropic** if $F(U) = 0$. The common dimension ω of all maximal totally isotropic subspaces of W is called the **Witt index** of (V, F), and hence maximal totally isotropic subspaces of V have dimension $M := \lambda + \omega$.

Isotropic subspaces of bounded height

The following theorem was proved by **Schlickewei (1985)** over $\mathbb{Q}$, by **Vaaler (1987)** over number fields, and by **Chan, F., Henshaw (2013)** over global function fields.

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Notice that in case $N = L$ and $M = 1$, this reduces to Cassels' theorem with the same exponent. Analogous result over $\overline{\mathbb{Q}}$, but with weaker exponents, was obtained by **F. (2008)** using a different technique.

Collections of isotropic subspaces

More generally, there exist collections of totally isotropic subspaces generating the full quadratic space. The following theorem was proved by **Schlickewei and Schmidt (1987)** over $\mathbb{Q}$ and by **Vaaler (1989)** over number fields.

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Theorem 2

Let (V, F) be an L-dimensional quadratic space in N variables over K and let $M \geq 1$ be the dimension of its maximal totally isotropic subspaces. There exists a collection of $L - M + 1$ maximal totally isotropic subspaces $U_0, \dots, U_{L-M} \subseteq V$ such that $V = \operatorname{span}_K \{U_0, \dots, U_{L-M}\}$, and for each $0 \leq i \leq L - M$,

$$H(U_0)H(U_i) \ll_{K,L,M} H(F)^{L-M} \mathcal{H}(V)^2.$$

Infinite family

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Theorem 3 (Chan, F., Henshaw (2010/2014))

Let (V, F) be an L-dimensional quadratic space in N variables over K of Witt index $\omega \geq 1$ and dimension of the radical $\lambda \geq 0$. There exists an infinite family of collections of maximal totally isotropic subspaces $\{U_{n1}, \ldots, U_{nJ}\}_{n=1}^{\infty} \subseteq V$, for an appropriately defined J, such that for each $n \geq 1$, $\mathrm{span}_K \{U_{n1}, \ldots, U_{nJ}\} = V$, and for each $1 \leq j \leq J$,

$$H(U_{nj}) \ll H(F)^{\mathrm{p}(L,\omega)} H(V)^{\mathrm{q}(\omega)},$$

where the constant in the upper bound depends on $K, N, L, \omega, \lambda, n$, and $\mathrm{p}(L,\omega)$, $\mathrm{q}(\omega)$ are polynomials: $\mathrm{p}(L,\omega)$ is linear in L, quartic in ω, and $\mathrm{q}(\omega)$ is cubic in ω. The constant depending on n is n^2 if K is a number field, e^{2n} if K is a function field, and 1 if $K = \overline{\mathbb{Q}}$.

Witt decomposition

Let K be a number field, global function field, or $\overline{\mathbb{Q}}$, and let (V, F) be an L-dimensional quadratic space in N variables over K. We use the same notation as above:

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A 2-dimensional subspace $\text{span}_K \{\mathbf{x}, \mathbf{y}\}$ of (V, F) is called a **hyperbolic plane** if $F(\mathbf{x}) = F(\mathbf{y}) = 0,\ F(\mathbf{x}, \mathbf{y}) \neq 0$.

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Theorem 4 (Witt decomposition theorem)

There exists an orthogonal decomposition of (V, F)

$$V = V^{\perp} \perp \mathbb{H}_1 \perp \ldots \perp \mathbb{H}_{\omega} \perp W,$$

where $\mathbb{H}_1, \ldots, \mathbb{H}_{\omega}$ *are hyperbolic planes and* W *is an anisotropic component.*

Effective Witt decomposition

It has been shown by **Vaaler (1989)** that over global fields

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Theorem 5 (F. (2007/2013))

There exists a Witt decomposition of the quadratic space (V, F) over a global field K such that

$$\max\{\mathcal{H}(\mathbb{H}_i), \mathcal{H}(W)\} \ll_{K,N,L,\omega} \left\{ H(F)^{\frac{2\omega+r}{4}} \mathcal{H}(V) \right\}^{\frac{(\omega+1)(\omega+2)}{2}}.$$

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Remark 4

An analogous result over $\overline{\mathbb{Q}}$, although with weaker bounds, has been obtained by **F. (2008)**.

Inhomogeneous quadratic polynomials

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Let $F(\mathbf{X})$ be an inhomogeneous quadratic polynomial in $N \geq 2$ variables with rational coefficients that has a nontrivial rational zero. Then it has such a zero $\mathbf{x}$ with

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The exponent in the upper bound is sharp.

The idea of the proof is to introduce an additional variable X_0 to homogenize F, and then apply Cassels' theorem while ensuring that $X_0 \neq 0$.

Missing subspaces

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A more general result of this kind was obtained recently. We state it over global fields, but there is also a version over $\overline{\mathbb{Q}}$ with weaker bounds. This result can be loosely interpreted as a distribution statement on small-height zeros of a quadratic form: they cannot be easily "cut out" by finite collections of polynomial maps.

Missing varieties

Theorem 7 (Chan, F., Henshaw (2013))

Let (V, F) be an isotropic quadratic space of dimension L in N variables over a global field K and let $M = \omega + \lambda \geq 1$ be the dimension of its maximal totally isotropic subspace. Let $\mathcal{Z}_K$ be a union of a finite collection of projective varieties defined over K such that F has a nontrivial zero in $V \not\subseteq \mathcal{Z}_K$, and let D be sum of degrees of these varieties. Then there exist M linearly independent zeros $\mathbf{x}_1, \dots, \mathbf{x}_M$ of F in $V \setminus \mathcal{Z}_K$ such that for each $1 \leq n \leq M$,

$$H(\mathbf{x}_n) \ll_{K,L,M,D} H(F)^{\frac{9L+11}{2}} \mathcal{H}(V)^{9L+12}.$$

Moreover, there exists a chain of totally isotropic subspaces $W_1 \subset \dots \subset W_M$ of (V, F) with $\dim_K W_n = n$ and $W_n \not\subseteq \mathcal{Z}_K$ for each $1 \leq n \leq M$ such that

$$\mathcal{H}(W_n) \ll_{K,L,M,D} H(F)^{15(L+1)-M} \mathcal{H}(V)^{27L+37}.$$

Other recent results

A great deal of work has been done over the years on different extensions and generalizations of Cassels' theorem. Here is a quick overview of a few more of our recent results:

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- Effective structure theorems for symplectic spaces over number fields, function fields, and their algebraic closures (**F. (2008)**).
- Small-height zeros of hermitian forms over positive definite quaternion algebras over totally real number fields (**Chan, F. (2010)**).
- Small-height zeros of hermitian forms missing zero sets of polynomials over positive definite quaternion algebras over totally real number fields (**F., Henshaw (2013)**).

Conclusion

An overview of this area can be found in my recent survey paper:

Heights and quadratic forms: on Cassels' theorem and its generalizations, in "Diophantine methods, lattices, and arithmetic theory of quadratic forms" (W. K. Chan, L. Fukshansky, R. Schulze-Pillot, and J. D. Vaaler, eds.), Contemporary Mathematics, AMS vol. 587 (2013), pg. 77–94

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