On distribution of well-rounded sublattices of $\mathbb{Z}^2$

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June 13, 2008

Introduction

Let $N \geq 2$ be an integer, and let $\Lambda \subseteq \mathbb{R}^N$ be a lattice of full rank. Define the **minimum** of Λ to be

$$|\Lambda| = \min_{\boldsymbol{x} \in \Lambda \setminus \{\boldsymbol{0}\}} \|\boldsymbol{x}\|,$$

where $\| \ \|$ stands for the usual Euclidean norm on $\mathbb{R}^N$. Let

$$S(\Lambda) = \{\boldsymbol{x} \in \Lambda : \|\boldsymbol{x}\| = |\Lambda|\}$$

be the set of *minimal vectors* of Λ. We say that Λ is a **well-rounded** lattice (abbreviated WR) if $S(\Lambda)$ spans $\mathbb{R}^N$.

WR lattices come up in connection with sphere packing, covering, and kissing number problems, coding theory, and the linear Diophantine problem of Frobenius, just to name a few of the contexts.

Still, the WR condition is special enough so that one would expect WR lattices to be rather sparce among all lattices.

McMullen's theorem

In 2005 C. McMullen showed that in a certain sense *unimodular* WR lattices are "well distributed" among all *unimodular* lattices in $\mathbb{R}^N$, where a unimodular lattice is a lattice with determinant equal to 1.

More specifically, he proved the following theorem, from which he derived the 6-dimensional case of the famous Minkowski's conjecture for unimodular lattices.

Theorem 1 (McMullen, 2005). *Let A be a subgroup of $SL_N(\mathbb{R})$ consisting of diagonal matrices with positive diagonal entries, and let Λ be a full-rank unimodular lattice in $\mathbb{R}^N$. If the closure of the orbit $A\Lambda$ is compact in the space of all full-rank unimodular lattices in $\mathbb{R}^N$, then it contains a WR lattice.*

Arithmetic problem

We consider an arithmetic problem: study the WR sublattices of $\mathbb{Z}^N$ and understand their distribution among all sublattices of $\mathbb{Z}^N$.

In this talk we describe our results for the case $N = 2$.

Question 1. *Which full-rank sublattices of $\mathbb{Z}^2$ are WR?*

Examples: WR sublattices of $\mathbb{Z}^2$:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mathbb{Z}^2, \quad \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mathbb{Z}^2$$

for any $a, b \in \mathbb{Z}$ - these come from ideals in $\mathbb{Z}[i]$ and have orthogonal bases.

No orthogonal basis:

$$\begin{pmatrix} 4 & 4 \\ 3 & -3 \end{pmatrix} \mathbb{Z}^2, \quad \begin{pmatrix} 7 & 7 \\ 5 & -5 \end{pmatrix} \mathbb{Z}^2, \quad \begin{pmatrix} 7 & -1 \\ 4 & 8 \end{pmatrix} \mathbb{Z}^2$$

Gauss's criterion

Lemma 2 (Gauss)**.** *Let* Λ *be a full-rank sublattice of* $\mathbb{Z}^2$*, let* $\boldsymbol{x}, \boldsymbol{y}$ *be a basis for* Λ*, and let* θ *be the angle between* $\boldsymbol{x}$ *and* $\boldsymbol{y}$*. If*

$$\frac{\pi}{3} \leq \theta \leq \frac{2\pi}{3}.$$

then the basis $\boldsymbol{x}, \boldsymbol{y}$ *contains a minimal vector of* Λ*.*

This leads to the following characterization of full-rank WR sublattices of $\mathbb{Z}^2$.

Lemma 3. *A sublattice* $\Lambda \subseteq \mathbb{Z}^2$ *of rank 2 is WR if and only if it has a basis* $\boldsymbol{x}, \boldsymbol{y}$ *with*

$$\|\boldsymbol{x}\| = \|\boldsymbol{y}\|, \ |\cos\theta| = \frac{|\boldsymbol{x}^t \boldsymbol{y}|}{\|\boldsymbol{x}\|\|\boldsymbol{y}\|} \leq \frac{1}{2}, \qquad (1)$$

where θ *is the angle between* $\boldsymbol{x}$ *and* $\boldsymbol{y}$*. Moreover, if this is the case, then the set of minimal vectors* $S(\Lambda) = \{\pm\boldsymbol{x}, \pm\boldsymbol{y}\}$*. In particular, a minimal basis for* Λ *is unique up to* $\pm$ *signs and reordering. Hence, if we require* θ *to lie in* $[\pi/3, \pi/2]$*, then it is uniquely determined by the lattice* Λ*; we will denote it by* $\theta(\Lambda)$*.*

Determinant and minima sets

Let

$$\mathrm{WR}(\mathbb{Z}^2) = \left\{ \Lambda \subseteq \mathbb{Z}^2 \ : \ \mathrm{rk}(\Lambda) = 2,\ \Lambda \text{ is WR} \right\}.$$

We want to understand how WR lattices are distributed among all sublattices of $\mathbb{Z}^2$. We start by defining the **minima** and **determinant** sets of elements of $\mathrm{WR}(\mathbb{Z}^2)$.

Let

$$\begin{aligned} \mathfrak{M} &:= \left\{ \min_{\mathbf{0} \neq \boldsymbol{x} \in \Lambda} \|\boldsymbol{x}\|^2 : \Lambda \in \mathrm{WR}(\mathbb{Z}^2) \right\} \\ &= \{a^2 + b^2 : a, b \in \mathbb{Z}\} \setminus \{0\}. \end{aligned}$$

Let

$$\begin{aligned} \mathcal{D} &:= \{\det(\Lambda) : \Lambda \in \mathrm{WR}(\mathbb{Z}^2)\} \\ &= \Big\{ (a^2+b^2)cd \ : \ a, b \in \mathbb{Z}_{\geq 0},\ \max\{a,b\} > 0, \\ &\qquad\qquad c, d \in \mathbb{Z}_{>0},\ 1 \leq \frac{c}{d} \leq \sqrt{3} \Big\} \\ &= \mathfrak{M}\mathcal{B}, \end{aligned}$$

where

$$\mathcal{B} = \left\{ cd \ : \ c, d \in \mathbb{Z}_{>0}, \ 1 \le \frac{c}{d} \le \sqrt{3} \right\}. \qquad (2)$$

It is not difficult to show that

$$\frac{\sqrt{3}\ |\Lambda|^2}{2} \le \det(\Lambda) \le |\Lambda|^2$$

for every $\Lambda \in \mathrm{WR}(\mathbb{Z}^2)$.

A classical result of E. Landau (1908) implies that $\mathfrak{M}$ has asymptotic density 0 in $\mathbb{Z}$, i.e.

$$\lim_{M \to \infty} \frac{|\{m \in \mathfrak{M} : m \le M\}|}{M} = 0.$$

Theorem 4. *The set $\mathcal{B}$ as in (2) has positive lower density. More precisely*

$$\liminf_{M \to \infty} \frac{|\{u \in \mathcal{B} : u \le M\}|}{M} \ \ge \ \frac{3^{\frac{1}{4}} - 1}{2 \cdot 3^{\frac{1}{4}}} \approx 0.12008216\ldots$$

Since $1 \in \mathfrak{M}$, $\mathcal{B} \subseteq \mathcal{D}$, and we conclude that $\mathcal{D}$ also has positive lower density in $\mathbb{Z}_{>0}$.

Number of WR sublattices with fixed determinant

Question 2. *For a fixed* $u \in \mathcal{D}$*, how many lattices in* $\mathrm{WR}(\mathbb{Z}^2)$ *have determinant equal to* u*?*

For each $u \in \mathbb{Z}_{>0}$, let

$$\mathcal{N}(u) = |\{\Lambda \in \mathrm{WR}(\mathbb{Z}^2) : \det(\Lambda) = u\}|,$$

so $\mathcal{N}(u) \neq 0$ if and only if $u \in \mathcal{D}$.

We will give an explicit formula for $\mathcal{N}(u)$ and talk about its rate of growth. This provides information about the distribution of elements of $\mathrm{WR}(\mathbb{Z}^2)$ among all sublattices of $\mathbb{Z}^2$.

This problem can be interpreted in terms of counting integral points on varieties. Let us say that two points

$$\boldsymbol{x} = (x_1, x_2, x_3, x_4)^t, \ \boldsymbol{y} = (y_1, y_2, y_3, y_4)^t \in \mathbb{R}^4$$

are equivalent if for some $U \in GL_2(\mathbb{Z})$,

$$U \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix} = \begin{pmatrix} y_1 & y_3 \\ y_2 & y_4 \end{pmatrix}.$$

The number of *all* full-rank sublattices of $\mathbb{Z}^2$ with determinant equal to u, call it $F_2(u)$, is precisely the number of integral points on the hypersurface

$$x_1 x_4 - x_2 x_3 = u,$$

modulo this equivalence. This number is well known: if u has prime factorization

$$u = q_1^{c_1} \dots q_m^{c_m},$$

then

$$F_2(u) = \prod_{j=1}^{m} \frac{q_j^{c_j+1} - 1}{q_j - 1},$$

which grows linearly in u.

On the other hand, by Lemma 3, $\mathcal{N}(u)$, the number of *well-rounded* full-rank sublattices of $\mathbb{Z}^2$ with determinant equal to u, is the number of integral points on the subset of the variety

$$x_1x_4 - x_2x_3 = u,\ x_1^2 + x_2^2 - x_3^2 - x_4^2 = 0,$$

defined by the inequality

$$2|x_1x_3 + x_2x_4| \leq x_1^2 + x_2^2,$$

modulo the same equivalence. This makes direct counting much harder.

Arithmetic functions

To state an explicit formula for $\mathcal{N}(u)$, we need to introduce some additional notation.

For each $u \in \mathbb{Z}_{>0}$, define

$$\alpha(u) = \left|\left\{(a,b) \in \mathbb{Z}^2_{\geq 0} \ : \ a^2 + b^2 = u,\ a \leq b, \right.\right.$$
$$\left.\left. \mathrm{gcd}(a,b) = 1\right\}\right|,$$

if $u > 2$, and $\alpha(1) = \alpha(2) = \frac{1}{2}$.

Let

$$\beta(u) = \left|\left\{d \in \mathbb{Z}_{>0} : d \mid u \text{ and } \sqrt{\frac{u}{\sqrt{3}}} \leq d \leq \sqrt{u}\right\}\right|.$$

Also let

$$\delta_1(u) = \begin{cases} 1 & \text{if } u \text{ is a square} \\ 2 & \text{if } u \text{ is not a square,} \end{cases}$$

and

$$\delta_2(u) = \begin{cases} 0 & \text{if } u \text{ is odd} \\ 1 & \text{if } u \text{ is even, } \frac{u}{2} \text{ is a square} \\ 2 & \text{if } u \text{ is even, } \frac{u}{2} \text{ is not a square.} \end{cases}$$

Theorem 5. *Let $u \in \mathbb{Z}_{>0}$, and let $\mathcal{N}(u)$ be the number of lattices in* WR($\mathbb{Z}^2$) *with determinant equal to u. If $u = 1$ or 2, then $\mathcal{N}(u) = 1$, the corresponding lattice being either $\mathbb{Z}^2$ or $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbb{Z}^2$, respectively. Let $u > 2$, and define*

$$t = t(u) = \begin{cases} u & \text{if } u \text{ is odd} \\ \frac{u}{2} & \text{if } u \text{ is even.} \end{cases}$$

Then:

$$\begin{aligned} \mathcal{N}(u) &= \delta_1(t)\beta(t) + \delta_2(t)\beta\left(\frac{t}{2}\right) \\ &+ 4 \sum_{\substack{n|t, 1<n<t/2 \\ n \text{ not a square}}} \alpha\left(\frac{t}{n}\right)\beta(n) \\ &+ 2 \sum_{\substack{n|t, 1\leq n<t/2 \\ n \text{ a square}}} \alpha\left(\frac{t}{n}\right)(2\beta(n) - 1). \end{aligned}$$

In particular, if $u \notin \mathcal{D}$, then the right hand side of this formula is equal to zero.

Asymptotics

Corollary 6. *For each* $u \in \mathbb{Z}_{>0}$*,*

$$\mathcal{N}(u) \le O\left(\left(\frac{\sqrt{2}\log u}{\omega(u)}\right)^{2\omega(u)}\right),$$

where $\omega(u)$ *is the number of distinct prime divisors of* u*. Moreover,*

$$\mathcal{N}(u) < O\left((\log u)^{\log 8}\right),$$

for all $u \in \mathcal{D}$ *outside of a subset of asymptotic density 0. However, there exist infinite sequences* $\{u_k\}_{k=1}^{\infty} \subset \mathcal{D}$ *such that for every* $k \ge 1$

$$\mathcal{N}(u_k) \ge (\log u_k)^k.$$

For instance, there exists such a sequence with $u_k \le \exp\left(O(k(\log k)^2)\right)$ *and* $\omega(u_k) = O(k \log k)$*.*

Example of an extremal determinant sequence

Let $v_n = \prod_{i=1}^{n} p_i^2$, where $p_1, p_2, \ldots$ are primes congruent to 1 mod 4; by Dirichlet's theorem, there are infinitely many of them: for instance, the first 9 such primes are 5, 13, 17, 29, 37, 43, 47, 53, 61.

For each k choose the smallest n so that $v_n > (\log v_n)^k$, and let $u_k = v_n$ for this choice of n. Here is the actual data table for the first few values of the sequence $\{u_k\}$ computed with Maple.

k, n	$u_k = v_n$	$\mathcal{N}(u_k)$	$(\log u_k)^k$
1,2	4225	9	8.34877454
2,4	1026882025	518	430.5539044
3,7	5741913252704971225	215002	80589.79464
4,9	6001613673020239098038402 5	14324372	12413026.85

Let $\mathcal{N}_I(v_n)$ be the number of lattices in WR($\mathbb{Z}^2$) with determinant v_n coming from ideals in $\mathbb{Z}[i]$. For comparison with the table above,

$$\mathcal{N}_I(v_n) = 3^n.$$

Zeta function

Define **zeta-function of WR sublattices** of $\mathbb{Z}^2$ to be

$$\zeta_{\mathrm{WR}(\mathbb{Z}^2)}(s) = \sum_{\Lambda \in \mathrm{WR}(\mathbb{Z}^2)} (\det(\Lambda))^{-s} = \sum_{u=1}^{\infty} \mathcal{N}(u) u^{-s},$$

where $s \in \mathbb{C}$ is a complex variable.

Since $\mathcal{N}(u)$ can behave quite sporadically, it makes sense to study basic analytic properties of $\zeta_{\mathrm{WR}(\mathbb{Z}^2)}(s)$.

Theorem 7. *$\zeta_{\mathrm{WR}(\mathbb{Z}^2)}(s)$ has a pole of order two at $s = 1$ and is analytic for all $s \in \mathbb{C}$ with $\Re(s) > 1$.*

It should be remarked that in this talk the statement "Dirichlet series $\sum_{u=1}^{\infty} a_u u^{-s}$ has a pole of order μ at $s = s_0$" means only that

$$0 < \lim_{s \to s_0^+} |s - s_0|^{\mu} \sum_{u=1}^{\infty} |a_u u^{-s}| < \infty.$$

Theorem 7 can be viewed as a quantitive measure of the "size" of the set WR($\mathbb{Z}^2$) in the following sense.

Let $\mathcal{N}_I(u)$ be the number of ideals of norm u in $\mathbb{Z}[i]$, i.e. the number of *orthogonal* lattices of determinant u in WR($\mathbb{Z}^2$). As above, let $F_2(u)$ be the number of all full-rank sublattices of $\mathbb{Z}^2$ with determinant u. Clearly

$$\mathcal{N}_I(u) \leq \mathcal{N}(u) \leq F_2(u),$$

so $\zeta_{\mathrm{WR}(\mathbb{Z}^2)}(s)$ is "squeezed" between $\zeta_{\mathbb{Z}[i]}(s)$, Dedekind zeta function of Gaussian integers, and $\zeta_{\mathbb{Z}^2}(s) = \sum_{u=1}^{\infty} F_2(u) u^{-s}$.

$\zeta_{\mathbb{Z}[i]}(s)$ is analytic for all $s \in \mathbb{C}$ with $\Re(s) > 1$, and has a simple pole at $s = 1$.

$\zeta_{\mathbb{Z}^2}(s)$ is analytic for all $s \in \mathbb{C}$ with $\Re(s) > 2$, and has a simple pole at $s = 2$.

While it is obvious that $\mathcal{N}(u)$ is much "closer" to $\mathcal{N}_I(u)$ than to $F_2(u)$, Theorem 7 shows that $\mathcal{N}_I(u)$ and $\mathcal{N}(u)$ are still of different "orders of magnitude."

Lower bound

We now give an outline of the proof of Theorem 7. We do this by proving that $\zeta_{\mathsf{WR}(\mathbb{Z}^2)}(s)$ can be bounded from below and from above by Dirichlet series with poles of order two at $s = 1$ which are analytic for all $s \in \mathbb{C}$ with $\Re(s) > 1$.

We start with a lower bound. The first observation is that $\mathcal{N}(u)$ is roughly related to Dirichlet convolution of arithmetic functions α and β, in particular it can be shown that for all real s,

$$\sum_{u=1}^{\infty} \mathcal{N}(u)u^{-s} =$$
$$= O\left(\left(\sum_{u=1}^{\infty} \alpha(t)u^{-s}\right)\left(\sum_{u=1}^{\infty} \beta(t)u^{-s}\right)\right),$$

where

$$t = t(u) = \begin{cases} u & \text{if } u \text{ is odd} \\ \frac{u}{2} & \text{if } u \text{ is even,} \end{cases}$$

as in Theorem 5.

Lemma 8. *Let $t = t(u)$ be as in Theorem 5. The Dirichlet series $\sum_{u=1}^{\infty} \alpha(t)u^{-s}$ is absolutely convergent in the half-plane $\Re(s) > 1$ with a simple pole at $s = 1$. Moreover, when $\Re(s) > 1$ it has an Euler product expansion:*

$$\sum_{u=1}^{\infty} \frac{\alpha(t)}{u^s} = \frac{1}{2}\left(1 + \frac{1}{2^s} + \frac{1}{4^s}\right) \prod_{l \equiv 1 (\mathrm{mod}\, 4)} \frac{l^s + 1}{l^s - 1},$$

where the product is over primes l.

On the other hand, one can prove that

$$\sum_{u=1}^{\infty} \left|\beta(t)u^{-s}\right| \geq \left|\frac{1}{2^{s+1}}\right| \sum_{u=1}^{\infty} \left|\beta(u)u^{-s}\right|. \quad (3)$$

Next, let $\mathcal{B} \subseteq \mathcal{D}$ be as in (2), then $\beta(u) > 0$ for each $u \in \mathcal{B}$, so for all real s,

$$\sum_{u=1}^{\infty} \beta(u)u^{-s} \geq \sum_{u \in \mathcal{D}} \beta(u)u^{-s} \geq \sum_{u \in \mathcal{B}} u^{-s}. \quad (4)$$

Theorem 4 guarantees that the set $\mathcal{B}$ has positive lower density in $\mathbb{Z}_{>0}$. Recall that **Dirichlet lower density** of the set $\mathcal{B}$ is defined by

$$\liminf_{s \to 1^+} \frac{\sum_{u \in \mathcal{B}} u^{-s}}{\sum_{u \in \mathbb{Z}} u^{-s}}.$$

It is a well-known fact that Dirichlet lower density of a set is greater or equal than its lower density, therefore $\sum_{u \in \mathcal{B}} u^{-s}$ is analytic for all $s \in \mathbb{C}$ with $\Re(s) > 1$ and has a simple pole at $s = 1$. Combining this fact with (3), (4), and Lemma 8 proves that $\zeta_{\mathrm{WR}(\mathbb{Z}^2)}(s)$ has a pole of order **at least** 2 at $s = 1$.

Upper bound

Two lattices $\Lambda_1, \Lambda_2 \subseteq \mathbb{R}^2$ are said to be **similar** if there exists a matrix $A \in O_2(\mathbb{R})$, the group of 2×2 real orthogonal matrices, and a real constant α such that

$$\Lambda_1 = \alpha A \Lambda_2.$$

This is an equivalence relation, and the equivalence classes of lattices under this relation in $\mathbb{R}^2$ are called **similarity classes**.

It is easy to see that WR property is preserved under similarity, and we can talk about similarity classes of lattices in $\text{WR}(\mathbb{Z}^2)$, let us write $\mathcal{C}_2$ for the set of such similarity classes. Notice that two lattices $\Lambda_1, \Lambda_2 \in \text{WR}(\mathbb{Z}^2)$ are similar if and only if

$$\theta(\Lambda_1) = \theta(\Lambda_2),$$

in the notation of Lemma 3. Therefore a similarity class in $\mathcal{C}_2$ is uniquely determined by the angle $\theta(\Lambda) \in [\pi/3, \pi/2]$, where $\sin\theta(\Lambda)$, $\cos\theta(\Lambda)$ must be rational numbers.

Thus similarity classes in $\mathcal{C}_2$ are in bijective correspondence with primitive Pythagorean triples (PPTs) whose shortest leg is shorter than half of the hypothenuse.

Then we will write $C(p,q) \in \mathcal{C}_2$, where

$$\sqrt{q^2 - p^2} < p \leq q$$

are relatively prime non-negative integers,

$$\sqrt{q^2 - p^2} < q/2,$$

and $\sin\theta(\Lambda) = p/q$ for each $\Lambda \in C(p,q)$. In particular, $C(1,1)$ is the similarity class of orthogonal WR sublattices of $\mathbb{Z}^2$, which come from ideals in $\mathbb{Z}[i]$. Then:

$$\begin{aligned}\zeta_{\mathrm{WR}(\mathbb{Z}^2)}(s) &= \sum_{\Lambda \in \mathrm{WR}(\mathbb{Z}^2)} (\det(\Lambda))^{-s} \\ &= \sum_{C(p,q) \in \mathcal{C}_2} Z_{p,q}(s),\end{aligned}$$

where

$$Z_{p,q}(s) = \sum_{\Lambda \in C(p,q)} (\det(\Lambda))^{-s}.$$

Theorem 9. *For each* $C(p,q) \in \mathcal{C}_2$*,*

$$Z_{p,q}(s) = \frac{1}{p^s}\zeta_{\mathbb{Z}[i]}(s),$$

where $\zeta_{\mathbb{Z}[i]}(s)$ *is the Dedekind zeta function of Gaussian integers.*

Let

$$\begin{aligned} \mathfrak{P} \;=\; \{(p,t,q) \;:\; & p,t,q \in \mathbb{Z}_{>0},\ t < p < q, \\ & \gcd(p,t,q) = 1,\ p^2 + t^2 = q^2\}, \end{aligned}$$

be the set of *all* PPTs, and let

$$\mathcal{P} = \{(p,t,q) \in \mathfrak{P} \;:\; t < q/2\} \cup \{(1,0,1)\}.$$

Notice that we include $(1,0,1)$ in $\mathcal{P}$, and we always write p for the longest leg of a PPT.

Then Theorem 9 implies that

$$\zeta_{\mathrm{WR}(\mathbb{Z}^2)}(s) = \zeta_{\mathbb{Z}[i]}(s) \sum_{(p,t,q)\in\mathcal{P}} \frac{1}{p^s}.$$

Notice that for every $(p,t,q) \in \mathcal{P}$, we have

$$\sum_{(p,t,q)\in\mathcal{P}} \left|\frac{1}{p^s}\right| \le \left|\left(\frac{2}{\sqrt{3}}\right)^s\right| \sum_{(p,t,q)\in\mathcal{P}} \left|\frac{1}{q^s}\right|,$$

since $\frac{\sqrt{3}}{2}q \le p \le q$.

Therefore:

$$\begin{aligned}\sum_{(p,t,q)\in\mathcal{P}} \left|\frac{1}{p^s}\right| &\le \left|\left(\frac{2}{\sqrt{3}}\right)^s\right| \sum_{(p,t,q)\in\mathcal{P}} \left|\frac{1}{q^s}\right| \\ &\le \left|\left(\frac{2}{\sqrt{3}}\right)^s\right| \sum_{(p,t,q)\in\mathfrak{P}} \left|\frac{1}{q^s}\right|.\end{aligned}$$

In other words, for all real s,

$$\zeta_{\mathrm{WR}(\mathbb{Z}^2)}(s) \le \zeta_{\mathbb{Z}[i]}(s) \left(\frac{2}{\sqrt{3}}\right)^s \sum_{(p,t,q)\in\mathfrak{P}} \frac{1}{q^s}.$$

Lemma 10. *The Dirichlet series* $\sum_{(p,t,q)\in\mathfrak{P}} \frac{1}{q^s}$ *has a simple pole at* $s = 1$ *and is analytic for all* $s \in \mathbb{C}$ *with* $\Re(s) > 1$*. Moreover when* $\Re(s) > 1$*, it has an Euler product type expansion*

$$\sum_{(p,t,q)\in\mathfrak{P}} \frac{1}{q^s} = \frac{1}{2} \prod_{l\equiv 1 \pmod 4} \frac{l^s+1}{l^s-1},$$

where the product is over primes l*.*

Since $\zeta_{\mathbb{Z}[i]}(s)$ is analytic for all $s \in \mathbb{C}$ with $\Re(s) > 1$ and has a simple pole at $s = 1$, we conclude that $\zeta_{\mathrm{WR}(\mathbb{Z}^2)}(s)$ is **analytic** for all $s \in \mathbb{C}$ with $\Re(s) > 1$, and has a pole of order **at most** 2 at $s = 1$.

Together with the lower bound we produced above, this completes the outline of the proof of Theorem 7.

Parameterization by ideals in $\mathbb{Z}[i]$

The reason for the connection between zeta-functions $Z_{p,q}(s)$ of similarity classes $C(p,q) \in \mathcal{C}_2$ and $\zeta_{\mathbb{Z}[i]}(s)$, as demonstrated by Theorem 9, is the existence of parameterization of these similarity classes by ideals in $\mathbb{Z}[i]$.

More precisely, let us define a subset of $\mathbb{Z}^2$

$$\mathcal{A} = \Big\{(a,b) \in \mathbb{Z}^2 : 0 < b < a,\ \mathrm{gcd}(a,b) = 1,\ 2 \nmid (a+b),\ \text{and either}\ b < a < \sqrt{3}b,\ \text{or}\ (2+\sqrt{3})b < a\Big\},$$

and consider a corresponding subset of $C(1,1)$

$$C'(1,1) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mathbb{Z}^2 \in C(1,1) : (a,b) \in \mathcal{A} \right\}.$$

Theorem 11. *For each* $(p,t,q) \in \mathcal{P}$*, there exists a unique lattice*

$$\Omega = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mathbb{Z}^2 \in C'(1,1),$$

where $a, b \in \mathbb{Z}_{>0}$ *are given by*

$$p = \max\{a^2 - b^2, 2ab\}, \text{ and } q = a^2 + b^2, \quad (5)$$

such that $\Lambda \in C(p,q)$ *if and only if*

$$\Lambda = \operatorname{span}_{\mathbb{Z}} \left\{ \boldsymbol{x}, \begin{pmatrix} \frac{\sqrt{q^2-p^2}}{q} & -\frac{p}{q} \\ \frac{p}{q} & \frac{\sqrt{q^2-p^2}}{q} \end{pmatrix} \boldsymbol{x} \right\} \quad (6)$$

for some $\boldsymbol{x} \in \Omega$*. Moreover, every lattice in the set* $C'(1,1)$ *parametrizes some similarity class* $C(p,q)$ *with* p, q *as in (5) in this way.*

Structure of $\mathcal{C}_2$

Theorem 11 indicates that each similarity class $C(p,q)$ has algebraic structure of an ideal in $\mathbb{Z}[i]$.

In fact, the entire set of similarity classes $\mathcal{C}_2$ can be naturally endowed with an algebraic structure, which makes it reasonable to think of $\mathcal{C}_2$ as the **moduli space** of WR sublattices of $\mathbb{Z}^2$.

Let $A = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$, and $C = \begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix}$, all in $\mathrm{GL}_3(\mathbb{Z})$, and let

$$G = \langle I_3, A, B, C \rangle$$

be the non-commutative monoid generated by A, B, C with identity I_3, the 3×3 identity matrix. Let us think of elements of $\mathfrak{P}$ as vectors in $\mathbb{Z}^3$, and for each $M \in G$ define

$$M(x,y,z) = M \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \qquad (7)$$

It is a well known fact that for every $(x,y,z) \in \mathfrak{P}$, $A(x,y,z), B(x,y,z), C(x,y,z) \in \mathfrak{P}$. Moreover, every $(x,y,z) \in \mathfrak{P}$ can be obtained in a unique way by applying a sequence of linear transformations A, B, C to $(3,4,5)$, the smallest triple in $\mathfrak{P}$. Hence (7) defines a free action of G on $\mathfrak{P}$ by left multiplication. The set $\mathfrak{P}$ has the structure of an infinite rooted ternary tree with respect to this action, so G is a free monoid, and this tree is precisely the Cayley digraph of G with respect to the generating set $\{A, B, C\}$.

Question 3. *What subset of G corresponds to $\mathcal{P}$, and so to $\mathcal{C}_2$?*

Theorem 12. *The set $\mathcal{C}_2$ of similarity classes of lattices in* WR($\mathbb{Z}^2$) *has algebraic structure of an infinitely generated free non-commutative monoid with the class $C(1,1)$ of orthogonal well-rounded lattices serving as identity. As a combinatorial object, $\mathcal{C}_2$ has the structure of a regular rooted infinite tree, where each vertex has infinite degree, which is precisely the Cayley digraph of this monoid.*

More specifically, as a monoid $\mathcal{C}_2$ is isomorphic to the following submonoid of G:

$$H = \Big\{A^2NB, A^kB, ABNB, C^2NB, C^kB, CBNB \\ (AC)^kA^2NB, (AC)^kABNB, (AC)^kAB, \\ (CA)^kC^2NB, (CA)^kCBNB, (CA)^kCB \ : \\ N \in G, k \in \mathbb{Z}_{>0}\Big\}.$$

This isomorphism is given by the mapping

$$C(p,q) \to M \in H \ : \ M(1,0,1) = (p, \sqrt{q^2 - p^2}, q),$$

let us denote this corresponding element M of H by $M_{p,q}$. Then multiplication on $\mathcal{C}_2$ is defined by

$$C(p_1,q_1) * C(p_2,q_2) = M_{p_1,q_1} M_{p_2,q_2},$$

where on the left we have matrix multiplication. This operation is clearly non-commutative, and $C(1,1)$ corresponds to $M_{1,1} = I_3$, the identity.

Distribution among all WR lattices in $\mathbb{R}^2$

So far we have been discussing questions of distribution of lattices in $\mathrm{WR}(\mathbb{Z}^2)$ among all lattices in $\mathbb{Z}^2$. Another interesting question is:

Question 4. *How are lattices in* $\mathrm{WR}(\mathbb{Z}^2)$ *distributed among all WR lattices in* $\mathbb{R}^2$*?*

To address this question, let us introduce some notation. For each lattice Λ in $\mathbb{R}^2$, let us write

$$\langle \Lambda \rangle = \{\alpha U \Lambda : \alpha \in \mathbb{R}_{>0},\ U \in O_2(\mathbb{R})\}$$

for its similarity class. Let

$$\mathrm{Sim}(\mathbb{R}^2) = \{\langle \Lambda \rangle \ :\ \Lambda \text{ is WR}\}.$$

Recall that

$$\langle \Lambda_1 \rangle = \langle \Lambda_2 \rangle \text{ iff } \theta(\Lambda_1) = \theta(\Lambda_2).$$

Then we can define a metric on the space $\mathrm{Sim}(\mathbb{R}^2)$:

$$d_s(\Lambda_1, \Lambda_2) = \left| \sin\theta(\Lambda_1) - \sin\theta(\Lambda_2) \right|.$$

Notice that if $\Lambda \in C(p,q)$, where $C(p,q) \in \mathcal{C}_2$, then

$$C(p,q) = \langle \Lambda \rangle \cap \mathsf{WR}(\mathbb{Z}^2),$$

and

$$\sin\theta(\Lambda) = \frac{p}{q} \in \left(\frac{\sqrt{3}}{2}, 1\right].$$

By abuse of notation (when no confusion can arise), let us identify $C(p,q)$ with $\langle \Lambda \rangle$ and $\mathcal{C}_2$ with the set

$$\{\langle \Lambda \rangle \ : \ \Lambda \in C(p,q) \text{ for some } C(p,q) \in \mathcal{C}_2\}.$$

For any lattice $\Lambda \in \mathsf{Sim}(\mathbb{R}^2)$ and $C(p,q) \in \mathcal{C}_2$, we can also define

$$d_s(C(p,q), \Lambda) := d_s(\Lambda_{p,q}, \Lambda),$$

where $\Lambda_{p,q}$ is any lattice from $C(p,q)$. We then have the following result.

Theorem 13. *The set $\mathcal{C}_2$ is dense in the set $\mathsf{Sim}(\mathbb{R}^2)$ with respect to the metric d_s. Moreover, for every $\Lambda \in \mathsf{Sim}(\mathbb{R}^2)$, there exist infinitely many $C(p,q) \in \mathcal{C}_2$ such that*

$$d_s(\mathcal{C}(p,q), \Lambda) \le \frac{2\sqrt{2}}{q}.$$