Height bounds on zeros of quadratic forms over $\overline{\mathbb{Q}}$

Lenny Fukshansky
Claremont McKenna College
(joint work with W. K. Chan and G. Henshaw)

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Cassels' Theorem

Let $N \geq 2$, and let $F(\mathbf{X})$ be an integral quadratic form in N variables. A famous result of **J. W. S. Cassels** (1955) states that if F is isotropic over $\mathbb{Q}$, then it has a nontrivial integral zero $\mathbf{x}$ such that

$$|\mathbf{x}| \ll_N |F|^{\frac{N-1}{2}}, \tag{1}$$

where

$$|\mathbf{x}| := \max_{1 \leq i \leq N} |x_i|, \quad |F| := \max_{1 \leq i,j \leq N} |f_{ij}|,$$

and the constant in the upper bound is explicit. The exponent $\frac{N-1}{2}$ in the upper bound is best possible, as shown by an example due to **M. Knesser**.

Over more general fields

Analogues of Cassels' theorem over a global field K with respect to appropriately defined height functions have been obtained:

- In 1975 by **S. Raghavan** when K is a number field
- In 1987 by **A. Prestel** when K is a rational function field
- In 1997 by **A. Pfister** when K is an algebraic function field

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The exponent on height of F in the upper bounds is again $\frac{N-1}{2}$. There have been many extensions and generalizations of Cassels' theorem over the years. An overview of these is presented in my recent survey paper:

Heights and quadratic forms: on Cassels' theorem and its generalizations, in "Diophantine methods, lattices, and arithmetic theory of quadratic forms" (W. K. Chan, L. Fukshansky, R. Schulze-Pillot, and J. D. Vaaler, eds.), Contemporary Mathematics, AMS vol. 587 (2013), pg. 77–94

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Over $\overline{\mathbb{Q}}$, some such results turn out to be possible. This is the subject of the current talk.

Notation and absolute values

- K = a number field
- $d = [K : \mathbb{Q}]$ = global degree of K over $\mathbb{Q}$
- $M(K)$ = set of all places of K
- $\forall\ v \in M(K)$, K_v = completion of K at v, $d_v = [K_v : \mathbb{Q}_v]$
- $\forall\ u \in M(\mathbb{Q})$, $\sum_{v \in M(K), v|u} d_v = d$
- We choose absolute values $\forall\ v \in M(K)$ so that the **product formula** holds in the following form:

$$\prod_{v \in M(K)} |a|_v^{d_v} = 1,$$

for all $0 \neq a \in K$.

Heights

Let $N \geq 1$. For each $v \in M(K)$, define the local sup-norm

$$|\mathbf{x}|_v = \max_{1 \leq i \leq N} |x_i|_v$$

on K_v^N, and for each $v \mid \infty$ also define the local L_2-norm

$$\|\mathbf{x}\|_v = \left(\sum_{i=1}^{N} |x_i|_v^2 \right)^{1/2}.$$

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Define **projective height function** H on K^N by

$$H(\mathbf{x}) = \prod_{v \in M(K)} |\mathbf{x}|_v^{d_v/d}$$

for each $\mathbf{x} \in K^N$. H is defined on the projective space $\mathbb{P}(K^N)$, since

$$H(a\mathbf{x}) = H(\mathbf{x}),\ \forall\ \mathbf{x} \in K^N,\ 0 \neq a \in K,$$

by the product formula.

Heights

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Clearly,

$$H(\mathbf{x}) \leq h(\mathbf{x})$$

for all $\mathbf{x} \in K^N$.

Schmidt height on subspaces

Let $V \subseteq K^N$ be an L-dimensional subspace, $1 \leq L \leq N$, and let $\mathbf{x}_1, \dots, \mathbf{x}_L$ be a basis for V.

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under the standard embedding. Define

$$\mathcal{H}(V) := \prod_{v \nmid \infty} |\mathbf{y}|_v^{d_v/d} \times \prod_{v \mid \infty} \|\mathbf{y}\|_v^{d_v/d}.$$

This definition is independent of the choice of the basis by the product formula. It was first given by **W. M. Schmidt (1967)**.

A couple remarks

The normalizing exponent $1/d$ in the definition of H and $\mathcal{H}$ ensures that all our heights are **absolute**, i.e. do not depend on the field of definition (in other words, defined over $\overline{\mathbb{Q}}$).

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Given $\mathbf{0} \neq \mathbf{x} = (x_1, \dots, x_N) \in \overline{\mathbb{Q}}^N$, let us write $[\mathbf{x}]$ for the corresponding projective point. For a fixed number field K, define

$$\deg_K \mathbf{x} := [K(x_1, \dots, x_N) : K],$$

$$\deg_K [\mathbf{x}] := \min\{\deg_K \mathbf{y} : \mathbf{y} \in \overline{\mathbb{Q}}^N, [\mathbf{y}] = [\mathbf{x}]\}.$$

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Northcott's property: Let $C \in \mathbb{R}_{>0}$, $D \in \mathbb{Z}_{>0}$, K a fixed number field. Then the sets

$$\left\{\mathbf{x} \in \overline{\mathbb{Q}}^N : \deg_K \mathbf{x} \leq D, h(\mathbf{x}) \leq C\right\},$$

$$\left\{[\mathbf{x}] \in \mathbb{P}(\overline{\mathbb{Q}}^N) : \deg_K [\mathbf{x}] \leq D, H(\mathbf{x}) \leq C\right\}$$

are finite.

Analogue of Cassels' theorem over $\overline{\mathbb{Q}}$

While Northcott's property for heights no longer holds over $\overline{\mathbb{Q}}$ when degree of points is not bounded, many problems naturally become easier in the absolute setting. For instance, a version of the original Cassels' theorem can be proved in a much simpler way with a considerably better bound over $\overline{\mathbb{Q}}$.

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Lemma 1

Let $2 \leq L \leq N$ and $V \subseteq \overline{\mathbb{Q}}^N$ an L-dimensional subspace. Let $F(\mathbf{X})$ be a quadratic form in N variables over $\overline{\mathbb{Q}}$. There exists $\mathbf{0} \neq \mathbf{y} \in V$ such that $F(\mathbf{y}) = 0$ and

$$h(\mathbf{y}) \leq 8 \times 3^{2(L-1)} \mathcal{H}(V)^{\frac{4}{L}} H(F)^{\frac{1}{2}}.$$

System of quadratic forms over $\overline{\mathbb{Q}}$

Theorem 2

Let $k \geq 2$ be an integer, $F_1, \ldots, F_k$ be quadratic forms in N variables over $\overline{\mathbb{Q}}$, and let $V \subseteq \overline{\mathbb{Q}}^N$ be an L-dimensional subspace, $N \geq L \geq \frac{k(k+1)}{2} + 1$. There exists $\mathbf{0} \neq \mathbf{w} \in V$ such that $F_m(\mathbf{w}) = 0$ for all $1 \leq m \leq k$ and

$$h(\mathbf{w}) \leq \left(3^{\frac{L^2}{2}} N^{\frac{3(L+1)}{2}} \mathcal{H}(V)\right)^{20B_k^2/81} \left(\prod_{n=1}^{k-1} H(F_n)\right)^{B_k} H(F_k)^2,$$

where $B_2 = 9$ and

$$B_k = \frac{1}{4} \times 36^{2^{k-2}} \prod_{m=3}^{k} m^{2^{k-m+1}}$$

for all $k \geq 3$.

Inhomogeneous polynomials over $\overline{\mathbb{Q}}$

Theorem 3

Let F and G be quadratic polynomials in $N \geq 4$ variables over $\overline{\mathbb{Q}}$, possibly inhomogeneous. Let m be an integer, $0 \leq m \leq N-4$, and $\mathcal{L}_1, \dots, \mathcal{L}_m$ be linear polynomials in N variables over $\overline{\mathbb{Q}}$, possibly inhomogeneous; the case $m = 0$ just means that there are no linear polynomials. Suppose that the system

$$F(\mathbf{x}) = G(\mathbf{x}) = \mathcal{L}_1(\mathbf{x}) = \cdots = \mathcal{L}_m(\mathbf{x}) = 0$$

has a nontrivial solution over $\overline{\mathbb{Q}}$. Then there exists a point $\mathbf{0} \neq \mathbf{y} \in \overline{\mathbb{Q}}^N$ such that $F(\mathbf{y}) = \mathcal{L}_1(\mathbf{y}) = \cdots = \mathcal{L}_m(\mathbf{y}) = 0$ and

$$h(\mathbf{y}) \leq 8(N+1)^{2m} 3^{2(N-m+1)(N-m)} H(F)^{\frac{1}{2}} \prod_{i=1}^{m} H(\mathcal{L}_i)^4.$$

Inhomogeneous polynomials over $\overline{\mathbb{Q}}$

Theorem 3, continuation

There also exists a point $\mathbf{0} \neq \mathbf{w} \in \overline{\mathbb{Q}}^N$ *such that*

$$F(\mathbf{w}) = G(\mathbf{w}) = \mathcal{L}_1(\mathbf{w}) = \cdots = \mathcal{L}_m(\mathbf{w}) = 0$$

and

$$h(\mathbf{w}) \leq \mathcal{M}(m, N) H(F)^{58} H(G)^3 \prod_{i=1}^{m} H(\mathcal{L}_i)^{180},$$

where

$$\mathcal{M}(m, N) = 18 \times 8^{38} (N+1)^{90m+8} (N+1-m)^{36} 3^{90(N-m+1)(N-m)}.$$

Over a fixed number field

Our method can also be used to obtain analogues of Theorems 2 and 3 with points in question having bounded degree over a fixed number field. By Northcott's property, this provides actual search bounds for zeros of systems of quadratic and linear equations as above. On the other hand, the bounds on height we can obtain this way are weaker.

Thank you!