Math in the Grocery Aisle: from stacking oranges to constructing error-correcting codes

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What is the best way to pack oranges?

(from Wikipedia)

How do you stack perfectly round oranges of equal size so that they take up the least amount of space?

In other words, what is the **densest** packing of oranges in a box?

Packing density

Each orange of radius R has volume $\frac{4\pi R^3}{3}$.

A cubic box with side length L has volume L^3.

If N equal oranges of radius R are packed into a cubic box with side length L, then the **proportion** of space in the box occupied by the oranges is

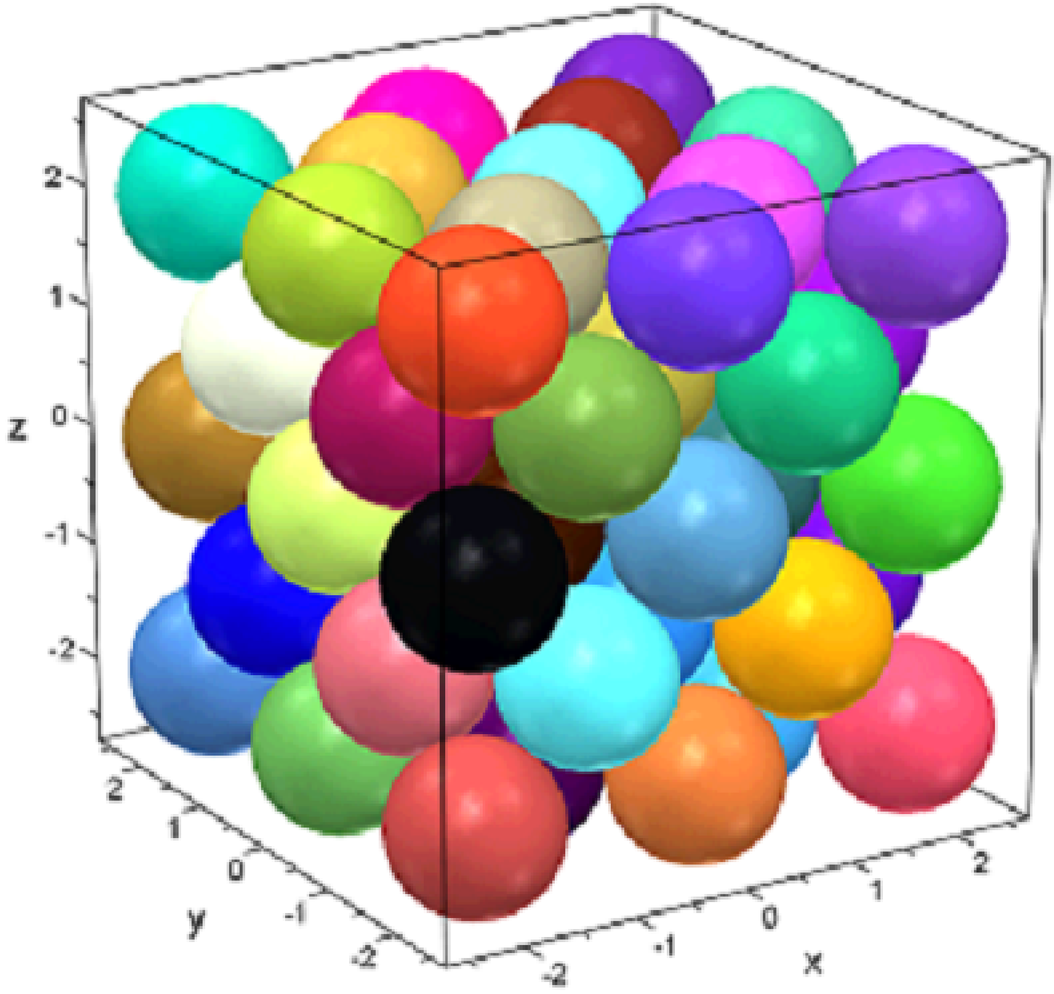
$$\frac{\text{Total volume of oranges}}{\text{Volume of the box}} = \frac{4\pi R^3 N}{3L^3}.$$

The larger is L, the larger is N. Then the **density** of an arrangement of oranges is the value of this ratio as L becomes very large, that is as the box grows unboundedly.

What arrangement of oranges gives you highest possible density? What is this highest possible density?

The big conjecture

The highest possible density $\approx 74\%$, which is achieved by the face-centered cubic packing:



FCC packing: mathPAD Online, vol. 15 (2006)

Historical note

From **Wikipedia**:

The conjecture is named after **Johannes Kepler**, *who stated the conjecture in* **1611** *in Strena sue de nive sexangula (On the Six-Cornered Snowflake). Kepler had started to study arrangements of spheres as a result of his correspondence with the English mathematician and astronomer* **Thomas Harriot** *in 1606. Harriot was a friend and assistant of* **Sir Walter Raleigh**, *who had set Harriot the problem of determining how best to stack cannon balls on the decks of his ships. Harriot published a study of various stacking patterns in 1591, and went on to develop an early version of atomic theory.*

Brief history

- In 1831, C. F. Gauss proved that the FCC packing gives optimal packing density among a large class of **periodic** (self-repeating) arrangements

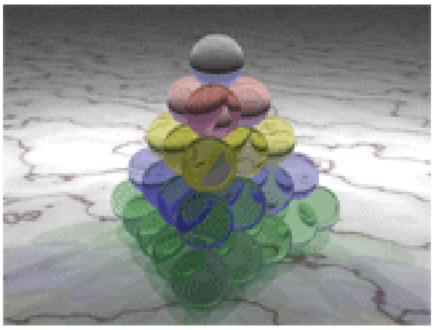
- In 1953, L. F. Toth showed that the proof of Kepler's conjecture can be reduced to a finite (albeit very large) number of computations

- **"Symmetric Bilinear Forms" by J. Milnor and D. Husemoller, 1973, p. 35:**

... according to [C. A.] Rogers, "many mathematicians believe and all physicists know that the density cannot exceed $\frac{\pi}{\sqrt{18}}$*"*

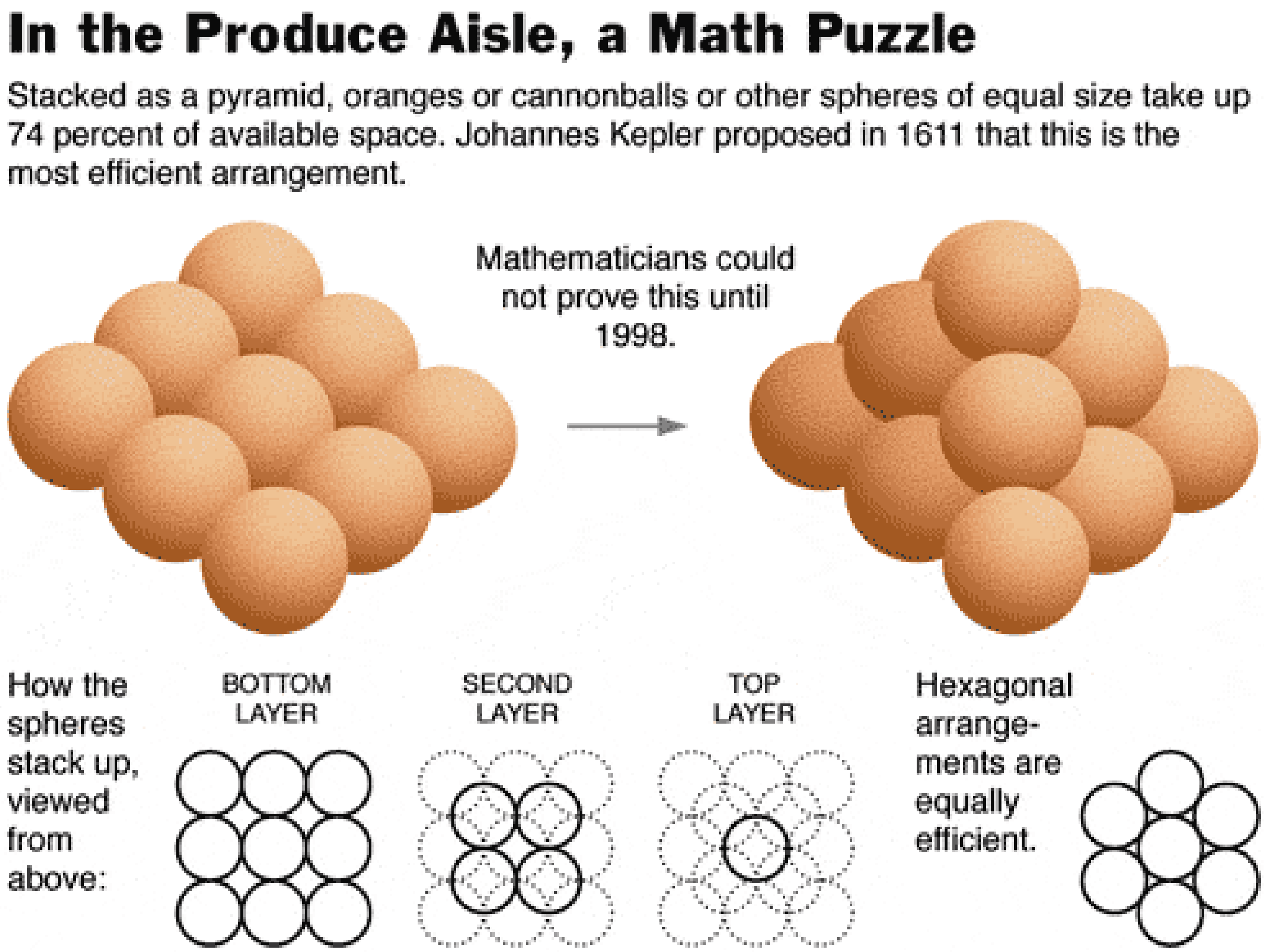
Done!

In 1998, T. C. Hales announced the proof, which was checked by a team of mathematicians, and finally published in 2005/2006 in the *Annals of Mathematics* (overview: 120 pages) and *Discrete and Computational Geometry* (full version: 265 pages); a part of it was done in collaboration with (Hales' graduate student at the time) S. P. Ferguson.



(from Wikipedia)

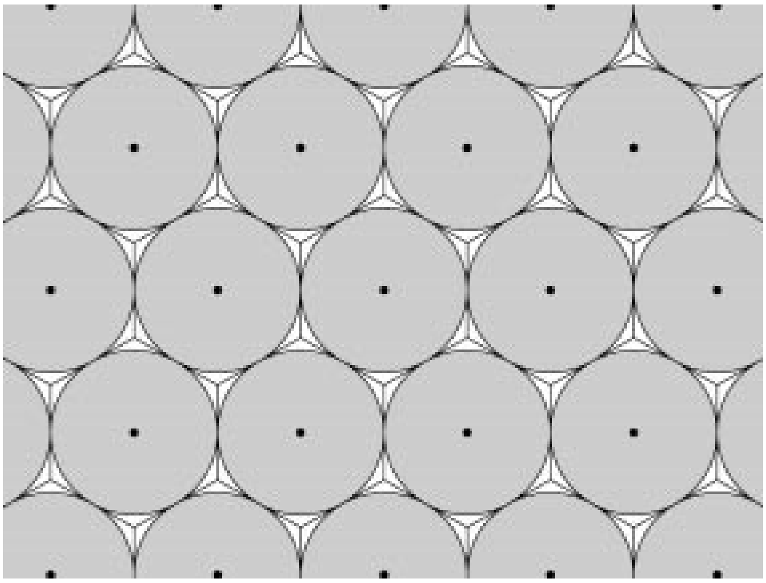
In the news



(from New York Times, April 6, 2004)

A similar problem in the plane

The two-dimensional analogue of Kepler's conjecture states that the best circle packing in the plane is the **hexagonal** arrangement, which gives density $\approx 90\%$:

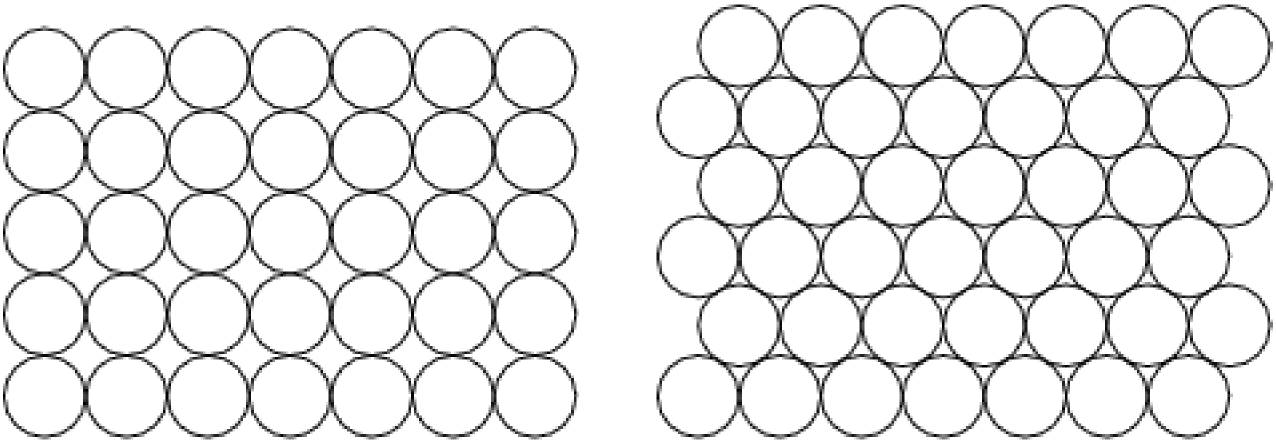


Lattices, Linear Codes, and Invariants, Part I, N. D. Elkies, AMS Notices, vol. 47 no. 10

This was proved by A. Thue in 1910 (a different proof was also given by L. F. Toth in 1940).

A comparison study

Here you can compare two different circle packing arrangements in the plane, and observe that the **hexagonal** is better than the **square**:



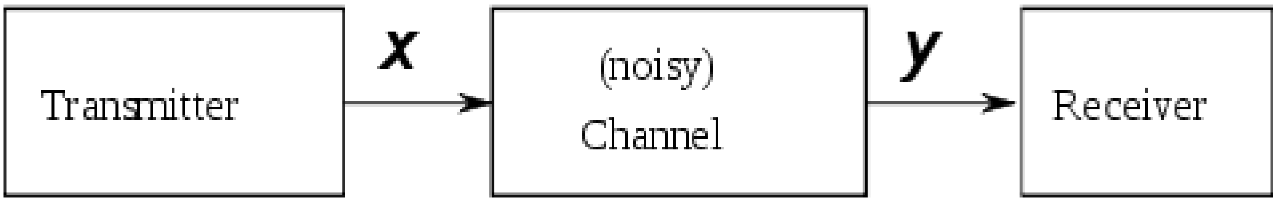
square packing *hexagonal packing*

(from MathWorld)

The gaps between the circles are bigger in the square packing, meaning that the hexagonal packing is denser.

An application: error-correcting codes

Suppose we want to transmit data in encoded format from a **transmitter** to a **receiver** over a **noisy channel**.



(from Wikipedia)

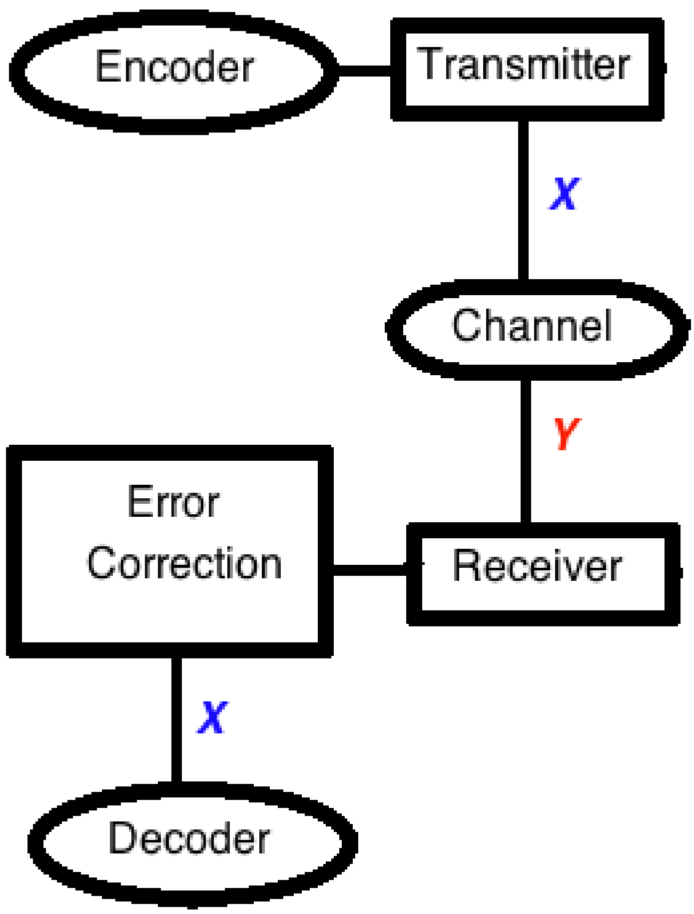
To transmit information, we want to **encode** it first using a collection of **codewords**, so that:

• The data is compressed to make the transmission **fast**.

• Our encoding allows the receiver to **self-correct** errors that happen in the channel.

Transmission with encoding

A more detailed picture of our transmission procedure looks like this:



Question: Why is this needed?

Is this used?

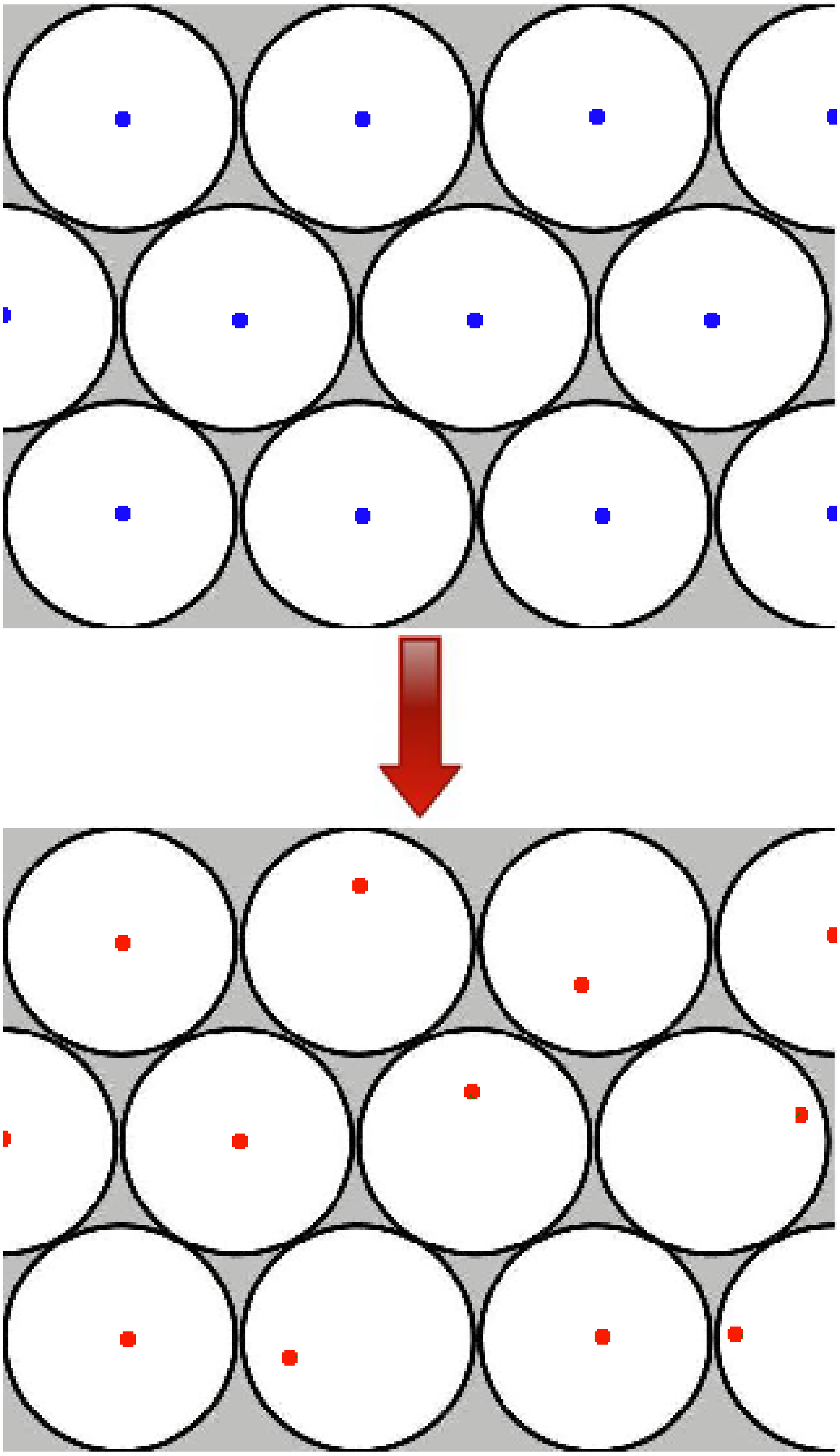
Error-correcting codes are used in:

- Telephone communications
- Cell phones
- Radio and TV transmission
- Recording and playing a CD
- Data transmission from satellite
- Compressing / storing data on a computer

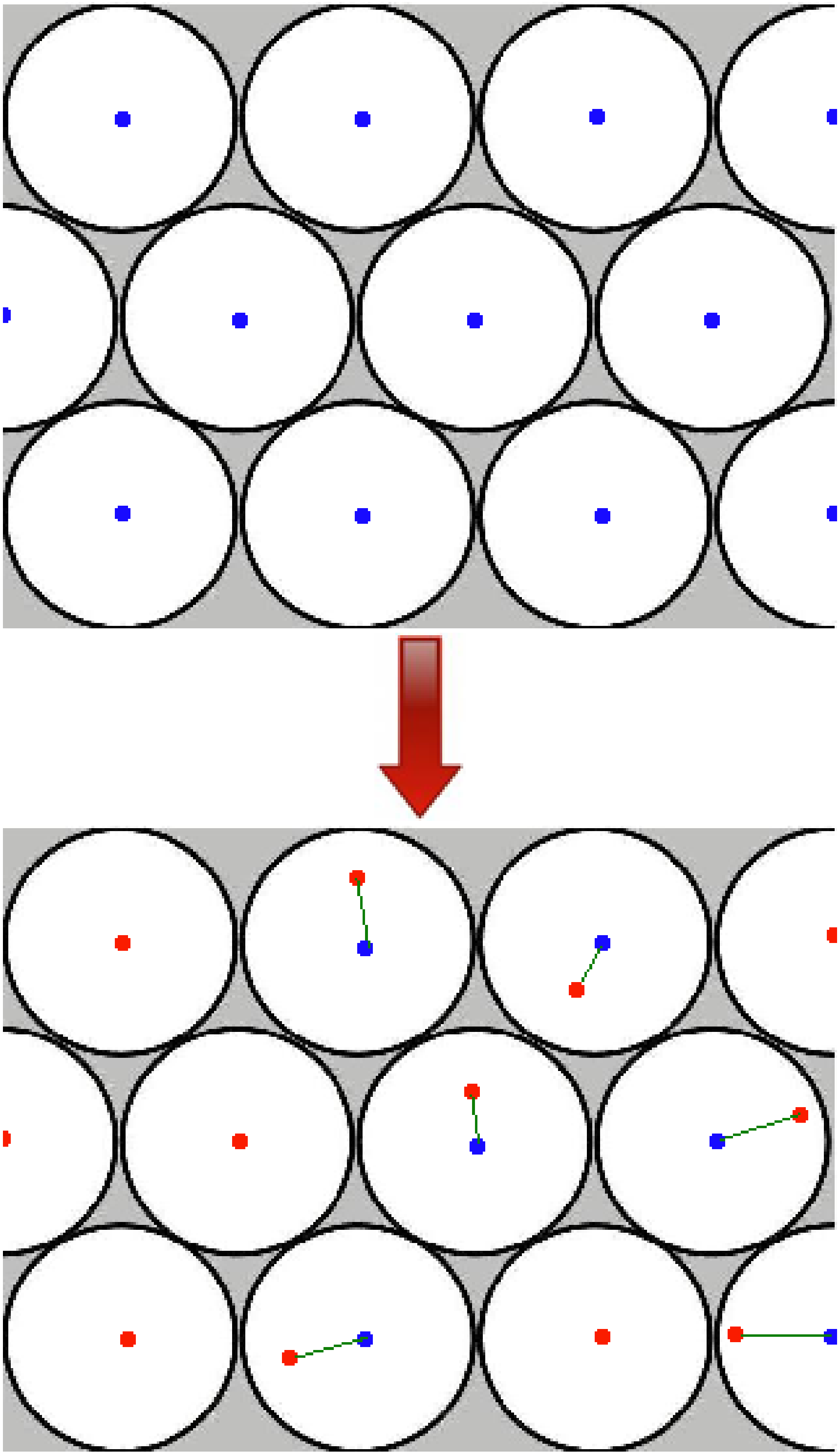
And many, many other engineering applications!

Question: How do we construct such codes?

A geometric idea



A geometric idea



Why circles?

Why do we use circles and not, for example, squares, triangles, or rectangular boxes?

The circle (and more generally, sphere in higher dimensions) has the simplest algebraic description, which turns out to be most convenient for the correction algorithm:

A circle of radius R centered at the point $P = (a, b)$ is the set of all points (x, y) in the plane, whose distance from P is at most R, namely

$$(x - a)^2 + (y - b)^2 \leq R^2.$$

This is very easy to use!

Does the arrangement of circles matter?

Each **codeword** in our code corresponds to the center of a circle in a packing arrangement.

The **size** of the code used in data transmission is the side length L of the square box, which fits all the circles corresponding to our codewords.

The **smaller** is L, the **faster** we can transmit the information.

The higher is **packing density** of our circle arrangement, the more circles we can fit into the box of the same size.

Therefore **efficient packing arrangements** allow to transmit more data in the same amount of time.

For even more efficient data transmission with error-correcting codes, **dense sphere packing arrangements** in higher dimensions can be used.