GEMS, FEBRUARY 28, 2009: SOLUTIONS HANDOUT

LENNY FUKSHANSKY

Abstract. These are solutions to the project handout.

1. Circle packing

Recall that $\pi = 3.14\ldots$ is an irrational number that is very important in mathematics. In particular, a circle of radius R has area equal to πR^2, so π is precisely the area of a unit circle (that is, a circle of radius one). Recall also that the area of a square of side-length L is L^2.

If N circles of radius R are packed into a square of side-length L, then the **density** of this packing is defined as

$$D = \frac{\text{Total area of all circles}}{\text{Area of the square}} = \frac{N\pi R^2}{L^2}.$$

When we say that the circles are **packed**, we mean that they do not overlap.

Problem 1. Show that for any packing of circles in a square, $0 < D \leq 1$. Do you think it is possible for $D = 1$? Why or why not?

Solution: A packing consists of at least one circle with non-zero area, which is why $D > 0$. On the other hand, the circles do not overlap and are contained inside of the square, therefore their combined total area is at most the area of the square, which is why $D \leq 1$. In fact, it is not possible for $D = 1$, since non-overlapping circles cannot be packed together without gaps between them, which means that not the entire area inside of the square will be covered by the circles.

Problem 2. Do you think it is possible for 16 circles of radius 4 to be packed into a square of side-length $16\sqrt{\pi}$? Why or why not?

Solution: If this was possible, then the density of this packing would be

$$D = \frac{16 \times \pi \times 4^2}{16^2 \pi} = 1,$$

which we know is not possible by our solution to Problem 1, so the answer is NO.

Problem 3. Suppose that P_1 is a packing of 10 circles of radius 2 into a square of side-length 20, and P_2 is a packing of 20 unit circles into a square of side-length $10\sqrt{\pi}$. Which packing has higher density? Explain.

Solution: Density of P_1 is

$$D_1 = \frac{10 \times \pi \times 2^2}{20^2} = \frac{\pi}{10} = 0.314\ldots,$$

and density of P_2 is

$$D_2 = \frac{20 \times \pi}{10^2 \pi} = \frac{1}{5} = 0.2.$$

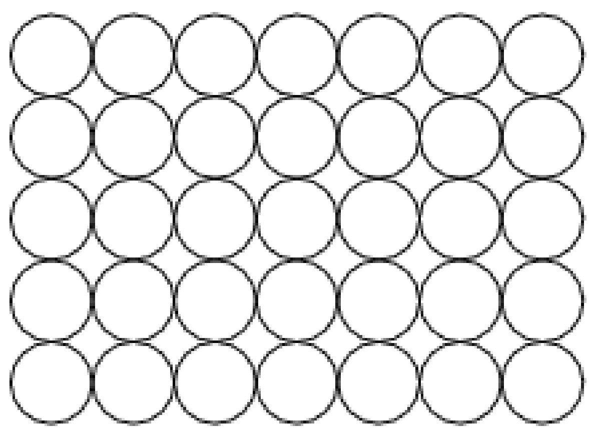
Therefore density of P_1 is higher than density of P_2.

Problem 4. If unit circles are packed into a square in such a way that precisely 10 of them touch one side of this square. What can you say about the side-length of this square? Explain.

Solution: If 10 unit circles touch one side of the square, then the length L of this side must be

$$L \geq 10 \times (\text{diameter of one unit circle}) = 10 \times 2 = 20.$$

Problem 5. Suppose that a **square arrangement** of unit circles



square packing

is packed into a square of side-length 20. How many circles are there? What is the density of this packing?

Solution: Since it is the square arrangement of circles as in the picture, the number of unit circles in a row along one side must be equal to the side-length of the square divided by the diameter of a unit circle, which is $20/2 = 10$. On the other hand, the number of such rows is also 10, hence the total number of circles is

$$(\text{number of rows}) \times (\text{number of circles in a row}) = 10 \times 10 = 100.$$

The density of this packing is

$$D = \frac{100 \times \pi}{20^2} = \frac{\pi}{4} = 0.785\ldots$$

2. Binary encoding

Here we talk about a method of encoding information using **binary codes**. Binary arithmetic is based on writing numbers to the base 2 instead of the usual decimal (base 10) system. This means that we are only allowed to use digits 0 and 1 to represent numbers. For example,

$$0 = 0,\ 1 = 1,\ 2 = 10,\ 3 = 11,\ 4 = 100,\ 5 = 101,\ 6 = 110, \dots \tag{1}$$

Problem 1. Continue the examples in (1) to represent all integers up to 26 in binary system.

Solution:

$$\begin{aligned}&7 = 111,\ 8 = 1000,\ 9 = 1001,\ 10 = 1010,\ 11 = 1011,\ 12 = 1100,\ 13 = 1101,\\ &14 = 1110,\ 15 = 1111,\ 16 = 10000,\ 17 = 10001,\ 18 = 10010,\ 19 = 10011,\\ &20 = 10100,\ 21 = 10101,\ 22 = 10110,\ 23 = 10111,\ 24 = 11000,\\ &25 = 11001,\ 26 = 11010.\end{aligned}$$

Problem 2. Can you now formulate a general principle? In other words, let A be a positive integer, written with digits $n_1, \dots, n_k$ in decimal system, that is

$$A = n_1 n_2 n_3 \dots n_k.$$

Can you explain how to write A in binary system?

Solution: If A is written with digits as $n_1 n_2 n_3 \dots n_k$, then

$$A = n_1 10^{k-1} + n_2 10^{k-2} + n_3 10^{k-3} + \dots + n_k 10^0.$$

We can write A in the same way as a sum of power of two with coefficients 0's and 1's, and then these coefficients are precisely the digits in the binary expansion of A. For example,

$$\begin{aligned}78 &= 7 \times 10^1 + 8 \times 10^0 = (2^2 + 2^1 + 1) \times (2^3 + 2^1) + 2^3\\ &= 2^5 + 2^3 + 2^4 + 2^2 + 2^3 + 2^1 + 2^3\\ &= 1 \times 2^6 + 0 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0,\end{aligned}$$

therefore in binary system $78 = 1001110$.

Problem 3. Now we can use binary system to do coding. Enumerate all the letters A through Z in English alphabet by numbers from 1 to 26, and use your findings from Problem 1 to convert them to binary system; for instance

$$A \to 1 \to 1,\ B \to 2 \to 10,\ C \to 3 \to 11, \dots$$

Continue this list to obtain the binary encoding for the entire alphabet.

Solution: Using (1) and our solution to Problem 1, we have:

$$A = 1,\ B = 10,\ C = 11,\ D = 100,\ E = 101,\ F = 110,$$
$$G = 111,\ H = 1000,\ I = 1001,\ J = 1010,\ K = 1011,\ L = 1100,\ M = 1101,$$
$$N = 1110,\ O = 1111,\ P = 10000,\ Q = 10001,\ R = 10010,\ S = 10011,$$
$$T = 10100,\ U = 10101,\ V = 10110,\ W = 10111,\ X = 11000,$$
$$Y = 11001,\ Z = 11010.$$

Problem 4. Use your conversion table from Problem 3 to write down binary encoding for the sentence "I LOVE MATH".

Solution: Here is the encoding word by word:

1001
1100 1111 10110 101
1101 1 10100 1000.

Problem 5. Suppose I send you an encoded binary message:

11 1111 100 1001 1110 111
1001 10011
110 10101 1110.

Can you decode it into words?

Solution: CODING IS FUN.

Remark. The binary code is used to store data in a computer, as well as in most other digital devices. Although just a simple binary code as we constructed here does not compress data to allow for fast transmission and does not do error-correction, there are more sophisticated variations of the binary code which do that; some of them also have real connections to sphere packing.

Department of Mathematics, 850 Columbia Avenue, Claremont McKenna College, Claremont, CA 91711
E-mail address: lenny@cmc.edu