A difference in numbers

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Claremont McKenna College

Gateway to Exploring Mathematical Sciences
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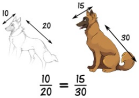
- To measure proportions: 

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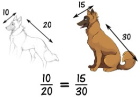
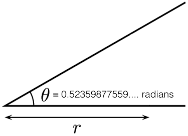
- To measure proportions: 

- To measure angles: 

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So what *are* the real numbers? (**L. Kronecker: "God created the integers; all else is the work of man"**)

Real numbers

One way to describe real numbers is in terms of a decimal expansion: a real number is a number of the form

$$\alpha = c_m \dots c_0 . b_1 b_2 b_3 \dots b_n \dots,$$

where $c_m, \dots, c_0, b_1, \dots, b_n, \dots$ are integers, called the **digits** of α.

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where $c_m, \dots, c_0, b_1, \dots, b_n, \dots$ are integers, called the **digits** of α. In other words:

$$\begin{aligned} \alpha &= c_m 10^m + \dots + c_0 10^0 + b_1 10^{-1} + \dots + b_n 10^{-n} + \dots \\ &= \sum_{k=0}^{m} c_k 10^k + \sum_{\ell=1}^{\dots} b_\ell 10^{-\ell}, \end{aligned}$$

where the second sum can be finite or infinite.

Decimal expansion

The representation

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Example 1

Both of these are rational:

$$\frac{129}{8} = 16.125 = 1 \times 10^1 + 6 \times 10^0 + 1 \times 10^{-1} + 2 \times 10^{-2} + 5 \times 10^{-3}$$

$$\frac{1}{3} = 0.333 \dots = 0 \times 10^0 + 3 \times 10^{-1} + 3 \times 10^{-2} + 3 \times 10^{-3} + \dots$$

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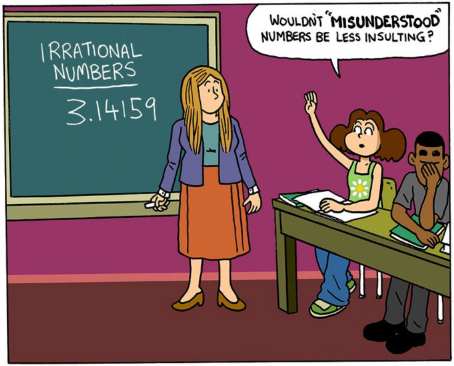

Rational vs Irrational numbers

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Example 2

Rational number: $\frac{3}{11} = 0.2727272727\ldots$
Irrational number: $\sqrt{2} = 1.41421356237\ldots$

Rational vs Irrational numbers

Then suppose we are building a real number by randomly picking out digits for its decimal expansion:

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237

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"Uh-oh ... we're outnumbered!"

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Perhaps there is some other description of real numbers, which does not depend on choosing a base?

Continued fractions

Every real number α can be written in the form

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \dots}}}},$$

where a_0 is an integer and $a_1, a_2, \dots$ are positive integers.

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$$\frac{7}{3} = 2 + \frac{1}{3}, \quad -\frac{93}{37} = -3 + \cfrac{1}{2 + \cfrac{1}{18}}, \quad \frac{103}{1647} = 0 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{102}}}.$$

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$$\pi = 3 + \frac{1}{7+\frac{1}{15+\frac{1}{1+\frac{1}{292+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{1+\frac{1}{3+\frac{1}{1+\dots}}}}}}}}}}}$$

Notation

We denote a continued fraction

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

by writing

$$\alpha = [a_0; a_1, a_2, a_3, \dots].$$

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Hence we have, for instance:

$$\begin{aligned} \frac{7}{3} &= [2; 3] \\ \sqrt{2} &= [1; 2, 2, 2, 2, \dots] \\ \pi &= [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, \dots] \end{aligned}$$

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i.e. we truncate the continued fraction for α at n-th step.

Then each such α_n is rational, so we can write it as $\alpha_n = \frac{p_n}{q_n}$ for some integers p_n and q_n. It is called the n-th **best rational approximation** to α to mean that it comes the closest to α among all rational numbers with denominators no bigger than q_n.

Best approximation: example

Let

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then:

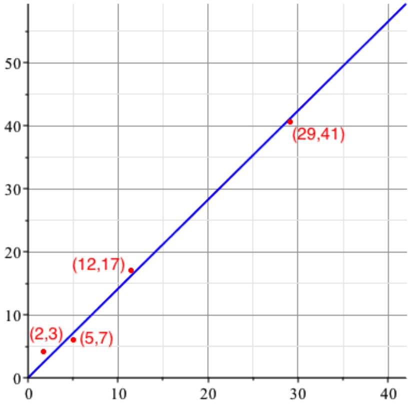
$$\alpha_1 = \frac{p_1}{q_1} = [1; 2] = 1 + \frac{1}{2} = \frac{3}{2} = 1.5$$

$$\alpha_2 = \frac{p_2}{q_2} = [1; 2, 2] = 1 + \frac{1}{2 + \frac{1}{2}} = \frac{7}{5} = 1.4$$

$$\alpha_3 = \frac{p_3}{q_3} = [1; 2, 2, 2] = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}} = \frac{17}{12} = 1.4166666666\dots$$

$$\alpha_4 = \frac{p_4}{q_4} = [1; 2, 2, 2, 2] = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}} = \frac{41}{29} = 1.41379310345$$

Geometrically...



$$y = \sqrt{2}x$$

Points with coordinates (q_n, p_n) are the closest integer points to this line with x-coordinate no bigger than q_n.

Gear ratios

Continued fractions have many fascinating applications, one of them to astronomy. Approximating the ratios of the periods of different planet orbits in the solar system to the Earth's year, one can build a model of the solar system with large circular gears.

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This was done by the Dutch physicist and astronomer Christiaan Huygens in 1680; he called his model the *automatic planetarium*.

For example, he calculated the period ratio for Mars to be

$$\alpha = \frac{197836}{105190},$$

and took its 5-th best continued fraction approximation

$$\alpha_5 = \frac{p_5}{q_5} = \frac{79}{42}.$$

He then used this information to design a pair of gears for Earth and Mars in his model with gear teeth ratio $158 : 84$.

Christian Huygens: automatic planetarium

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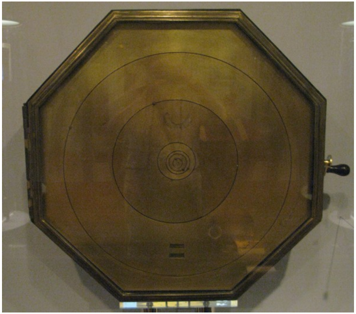

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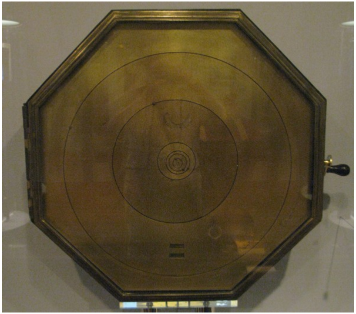

The Antikythera Mechanism

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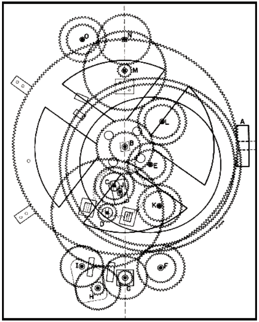
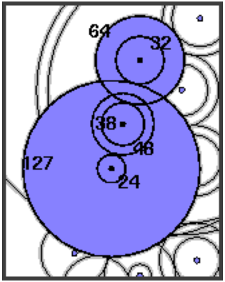


Ancient Greek **astronomical computer** dating to 1st century B.C. discovered in 1900 in the wreck of a cargo ship off the coast of Anthykhera, a small Greek island: it was a mechanical model of the solar system, capable of predicting the positions of the sun and moon in the zodiac on any given date.

The Sun-Moon Assembly: explanation by W. Casselman



The sun marker and the moon marker were driven by the two central gears (the moon axis threaded through the sun's), exactly like the hour and minute hands on a modern clock. The train of gears linking the sun's motion to that of the moon can be described by the meshing pattern and the numbers of teeth. The sun gear has 64 teeth. It meshes with the smaller of a 38,48 gear pair. The 48 meshes with the smaller of a 24,127 gear pair.

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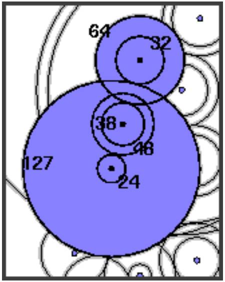

The 127 meshes with the 32 teeth of the moon gear. The ratio of angular speeds can then be calculated as

$$\frac{64}{38} \times \frac{48}{24} \times \frac{127}{32} = \frac{254}{19} = 13.36842\ldots,$$

which is the 5-th best continued fraction approximation to the astronomical ratio $13.368267\ldots$

Summary

- Any real number α has a **continued fraction** representation:

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}},$$

which we denote by writing $\alpha = [a_0; a_1, a_2, a_3, \dots]$.

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  which we denote by writing $\alpha = [a_0; a_1, a_2, a_3, \dots]$.
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- Continued fractions and their best approximations found use in many areas of science and engineering, e.g. in computing gear ratios for gear design in astronomy and related fields.

Thank you!