GEMS, NOVEMBER 4, 2017: PROJECT HANDOUT

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ABSTRACT. This project deals with the properties of a very famous irrational number, called the **golden ratio** and its connection to the famous **Fibonacci sequence**.

The **golden ratio** is defined as

$$\varphi = \frac{1+\sqrt{5}}{2}.$$

This number appears in numerous places in mathematics, architecture, engineering and science throughout history, starting with proportion calculations for particularly symmetric structures in ancient Egypt (e.g. Great Pyramid of Giza) and then ancient Greece and Rome.

Problem 1. Prove that

$$\varphi^2 = \varphi + 1,$$

and derive from it that

$$\varphi = 1 + \frac{1}{\varphi}. \tag{1}$$

Solution: Notice that

$$\begin{aligned} \varphi^2 &= \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{(1+\sqrt{5})^2}{4} = \frac{1+2\sqrt{5}+5}{4} \\ &= \frac{4+2(1+\sqrt{5})}{4} = 1 + \frac{1+\sqrt{5}}{2} = 1+\varphi\,. \end{aligned}$$

Dividing both sides of the equation $\varphi^2 = 1+\varphi$ by φ, we obtain (1).

Problem 2. Derive a continued fraction expansion for φ. Write it down in the

$$[a_0; a_1, a_2, \dots]$$

notation, and use it to prove that φ is irrational.
[*Hint:* Use formula (1), repeatedly substituting it into itself.]

Solution: Repeatedly using formula (1), we obtain:

$$\begin{aligned} \varphi &= 1 + \frac{1}{\varphi} = 1 + \frac{1}{1+\frac{1}{\varphi}} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{\varphi}}} \\ &= 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{\varphi}}}} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{\varphi}}}}} = \dots \end{aligned}$$

Therefore we see that

$$\varphi = [1; 1, 1, 1, \ldots],$$

which is an infinite continued fraction, hence defines an irrational number.

Problem 3. Let us write φ_n for the n-th continued fraction approximations to φ, i.e.

$$\varphi_n = [a_0; a_1, \ldots, a_n].$$

Compute φ_1 through φ_5 as fractions.

Solution:

$$\begin{aligned}
\varphi_1 &= [1;1] = 1 + \frac{1}{1} = 2,\\
\varphi_2 &= [1;1,1] = 1 + \cfrac{1}{1+\frac{1}{1}} = \frac{3}{2},\\
\varphi_3 &= [1;1,1,1] = 1 + \cfrac{1}{1+\frac{1}{1+\frac{1}{1}}} = \frac{5}{3},\\
\varphi_4 &= [1;1,1,1,1] = 1 + \cfrac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}} = \frac{8}{5},\\
\varphi_5 &= [1;1,1,1,1,1] = 1 + \cfrac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}}} = \frac{13}{8}.
\end{aligned}$$

Now let us define the **Fibonacci sequence**. Let

$$F_1 = 1, F_2 = 1,$$

and for every $n \geq 3$, let

$$F_n = F_{n-1} + F_{n-2}. \tag{2}$$

This sequence is named after the 12th century Italian mathematician Leonardo Fibonacci of Pisa, but its origins go back to the study of poetic structures in Sanskrit in ancient India. These numbers are so important in mathematics, science and engineering that there are many things named after them, including the mathematical journal *Fibonacci Quarterly* devoted entirely to the study of the Fibonacci sequence and its many connections.

Problem 4. Compute the first ten Fibonacci numbers.

Solution:

$$F_1 = 1, F_2 = 1, F_3 = F_2 + F_1 = 1 + 1 = 2, F_4 = F_3 + F_2 = 2 + 1 = 3,$$
$$F_5 = F_4 + F_3 = 3 + 2 = 5, F_6 = F_5 + F_4 = 5 + 3 = 8, F_7 = F_6 + F_5 = 8 + 5 = 13,$$
$$F_8 = F_7 + F_6 = 13 + 8 = 21, F_9 = F_8 + F_7 = 21 + 13 = 34, F_{10} = F_9 + F_8 = 34 + 21 = 55.$$

Problem 5. Now compute as fractions the ratios of the consecutive Fibonacci numbers

$$\frac{F_3}{F_2}, \frac{F_4}{F_3}, \frac{F_5}{F_4}, \frac{F_6}{F_5}, \frac{F_7}{F_6}.$$

Have you seen these numbers before?

Solution:
$$\frac{F_3}{F_2} = 2,\ \frac{F_4}{F_3} = \frac{3}{2},\ \frac{F_5}{F_4} = \frac{5}{3},\ \frac{F_6}{F_5} = \frac{8}{5},\ \frac{F_7}{F_6} = \frac{13}{8}.$$
These are precisely the numbers φ_1 through φ_5 from Problem 3.

Problem 6. Prove the general formula:
$$\varphi_n = \frac{F_{n+2}}{F_{n+1}},$$
for every $n \geq 1$.
[*Hint:* Use formula (2), repeatedly substituting it into itself.]

Solution: Repeatedly using formula (2), we obtain
$$\begin{aligned}
\frac{F_{n+2}}{F_{n+1}} &= \frac{F_{n+1}+F_n}{F_{n+1}} = 1 + \frac{F_n}{F_{n+1}} = 1 + \frac{1}{F_{n+1}/F_n} \\
&= 1 + \frac{1}{(F_n + F_{n-1})/F_n} = 1 + \frac{1}{1 + \frac{F_{n-1}}{F_n}} = 1 + \frac{1}{1 + \frac{1}{F_n/F_{n-1}}} \\
&= 1 + \frac{1}{1 + \frac{1}{(F_{n-1}+F_{n-2})/F_{n-1}}} = 1 + \frac{1}{1 + \frac{1}{1+\frac{F_{n-2}}{F_{n-1}}}} \\
&= 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{F_{n-1}/F_{n-2}}}} = \cdots = 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{\ddots + \frac{1}{1}}}} = \varphi_n\,.
\end{aligned}$$

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