Frame coherence and nearly orthogonal lattices

Lenny Fukshansky
Claremont McKenna College

(joint work with David Kogan, CGU)

Claremont ANTC Seminar
February 8, 2022

Frames and coherence

A spanning set $\mathcal{F} = \{\boldsymbol{f}_1, \dots, \boldsymbol{f}_k\} \subset \mathbb{R}^n$, $k \geq n$, of equal-norm vectors is called a **frame**.

Frames and coherence

A spanning set $\mathcal{F} = \{\boldsymbol{f}_1, \dots, \boldsymbol{f}_k\} \subset \mathbb{R}^n$, $k \geq n$, of equal-norm vectors is called a **frame**.

It is called a **unit frame** if $\|\boldsymbol{f}_j\| = 1$ for every $1 \leq j \leq k$.

Frames and coherence

A spanning set $\mathcal{F} = \{\boldsymbol{f}_1, \dots, \boldsymbol{f}_k\} \subset \mathbb{R}^n$, $k \geq n$, of equal-norm vectors is called a **frame**.

It is called a **unit frame** if $\|\boldsymbol{f}_j\| = 1$ for every $1 \leq j \leq k$.

It is called an **equiangular frame** if $|\langle \boldsymbol{f}_i, \boldsymbol{f}_j \rangle| = c$ for all $1 \leq i \neq j \leq k$, for some constant $c \in [0, 1]$.

Frames and coherence

A spanning set $\mathcal{F} = \{\boldsymbol{f}_1, \dots, \boldsymbol{f}_k\} \subset \mathbb{R}^n$, $k \geq n$, of equal-norm vectors is called a **frame**.

It is called a **unit frame** if $\|\boldsymbol{f}_j\| = 1$ for every $1 \leq j \leq k$.

It is called an **equiangular frame** if $|\langle \boldsymbol{f}_i, \boldsymbol{f}_j \rangle| = c$ for all $1 \leq i \neq j \leq k$, for some constant $c \in [0, 1]$.

Coherence of a frame $\mathcal{F}$ is defined as

$$C(\mathcal{F}) = \max_{1 \leq i \neq j \leq k} \frac{|\langle \boldsymbol{f}_i, \boldsymbol{f}_j \rangle|}{\|\boldsymbol{f}_i\| \|\boldsymbol{f}_j\|}.$$

Frame optimization

Frames are extensively used in coding theory, digital communications and data compression, among many other areas – they are important tools of Applied Harmonic Analysis, which generalize orthonormal bases.

Frame optimization

Frames are extensively used in coding theory, digital communications and data compression, among many other areas – they are important tools of Applied Harmonic Analysis, which generalize orthonormal bases.

- **Redundancy prevents loss of information:** maximize $k = |\mathcal{F}|$, the frame size $\Longleftrightarrow$ improve ability to correct erasures.

Frame optimization

Frames are extensively used in coding theory, digital communications and data compression, among many other areas – they are important tools of Applied Harmonic Analysis, which generalize orthonormal bases.

- **Redundancy prevents loss of information:** maximize $k = |\mathcal{F}|$, the frame size $\Longleftrightarrow$ improve ability to correct erasures.
- **Low coherence reduces signal overlap:** minimize coherence $\Longleftrightarrow$ reduce noise in transmission.

Frame optimization

Frames are extensively used in coding theory, digital communications and data compression, among many other areas – they are important tools of Applied Harmonic Analysis, which generalize orthonormal bases.

- **Redundancy prevents loss of information:** maximize $k = |\mathcal{F}|$, the frame size $\Longleftrightarrow$ improve ability to correct erasures.
- **Low coherence reduces signal overlap:** minimize coherence $\Longleftrightarrow$ reduce noise in transmission.

Welch bound:

$$C(\mathcal{F}) \geq \sqrt{\frac{k-n}{n(k-1)}},$$

where equality is achieved by equiangular frames. Cardinality of an equiangular frame in $\mathbb{R}^n$ is $\leq \frac{n(n+1)}{2}$, but usually linear in n.

Lattices: basic notions

A **lattice** $\Lambda \subset \mathbb{R}^n$ of full rank is a free $\mathbb{Z}$-module of rank n, which is the same as a discrete co-compact subgroup of $\mathbb{R}^n = \operatorname{span}_{\mathbb{R}} \Lambda$. Hence

$$\Lambda = \operatorname{span}_{\mathbb{Z}}\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\} = A\mathbb{Z}^n,$$

where $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n \in \mathbb{R}^n$ are $\mathbb{R}$-linearly independent **basis** vectors for Λ and $A = (\boldsymbol{a}_1 \ \dots \ \boldsymbol{a}_n)$ is the corresponding $n \times n$ basis matrix.

Lattices: basic notions

A **lattice** $\Lambda \subset \mathbb{R}^n$ of full rank is a free $\mathbb{Z}$-module of rank n, which is the same as a discrete co-compact subgroup of $\mathbb{R}^n = \operatorname{span}_{\mathbb{R}} \Lambda$. Hence

$$\Lambda = \operatorname{span}_{\mathbb{Z}}\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\} = A\mathbb{Z}^n,$$

where $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n \in \mathbb{R}^n$ are $\mathbb{R}$-linearly independent **basis** vectors for Λ and $A = (\boldsymbol{a}_1 \ \dots \ \boldsymbol{a}_n)$ is the corresponding $n \times n$ basis matrix.

The **determinant** of Λ is

$$\det \Lambda := \sqrt{\det(A^\top A)},$$

which is equal to the volume (quotient Lebesgue measure) of $\mathbb{R}^n/\Lambda$.

Lattices: minimal vectors

Minimal norm of a lattice Λ is

$$|\Lambda| = \min \left\{ \|\boldsymbol{x}\| : \boldsymbol{x} \in \Lambda \setminus \{\boldsymbol{0}\} \right\},$$

where $\| \ \|$ is Euclidean norm. The set of **minimal vectors** of Λ is

$$S(\Lambda) = \{\boldsymbol{x} \in \Lambda : \|\boldsymbol{x}\| = |\Lambda|\} = \{\pm \boldsymbol{x}_1, \dots, \pm \boldsymbol{x}_k\}.$$

Lattices: minimal vectors

Minimal norm of a lattice Λ is

$$|\Lambda| = \min\left\{\|\boldsymbol{x}\| : \boldsymbol{x} \in \Lambda \setminus \{\boldsymbol{0}\}\right\},$$

where $\| \ \|$ is Euclidean norm. The set of **minimal vectors** of Λ is

$$S(\Lambda) = \{\boldsymbol{x} \in \Lambda : \|\boldsymbol{x}\| = |\Lambda|\} = \{\pm \boldsymbol{x}_1, \dots, \pm \boldsymbol{x}_k\}.$$

- A lattice Λ is **well-rounded** (WR) if $\operatorname{span}_{\mathbb{R}} \Lambda = \operatorname{span}_{\mathbb{R}} S(\Lambda)$.

Lattices: minimal vectors

Minimal norm of a lattice Λ is

$$|\Lambda| = \min\{\|\boldsymbol{x}\| : \boldsymbol{x} \in \Lambda \setminus \{\boldsymbol{0}\}\},$$

where $\|\ \|$ is Euclidean norm. The set of **minimal vectors** of Λ is

$$S(\Lambda) = \{\boldsymbol{x} \in \Lambda : \|\boldsymbol{x}\| = |\Lambda|\} = \{\pm\boldsymbol{x}_1, \dots, \pm\boldsymbol{x}_k\}.$$

- A lattice Λ is **well-rounded** (WR) if $\operatorname{span}_{\mathbb{R}} \Lambda = \operatorname{span}_{\mathbb{R}} S(\Lambda)$.
- If $\operatorname{rk}\Lambda > 4$, a strictly stronger condition is that Λ is **generated by minimal vectors**, i.e. $\Lambda = \operatorname{span}_{\mathbb{Z}} S(\Lambda)$.

Lattices: minimal vectors

Minimal norm of a lattice Λ is

$$|\Lambda| = \min \{\|\boldsymbol{x}\| : \boldsymbol{x} \in \Lambda \setminus \{\boldsymbol{0}\}\},$$

where $\| \ \|$ is Euclidean norm. The set of **minimal vectors** of Λ is

$$S(\Lambda) = \{\boldsymbol{x} \in \Lambda : \|\boldsymbol{x}\| = |\Lambda|\} = \{\pm \boldsymbol{x}_1, \dots, \pm \boldsymbol{x}_k\}.$$

- A lattice Λ is **well-rounded** (WR) if $\operatorname{span}_{\mathbb{R}} \Lambda = \operatorname{span}_{\mathbb{R}} S(\Lambda)$.
- If $\operatorname{rk} \Lambda > 4$, a strictly stronger condition is that Λ is **generated by minimal vectors**, i.e. $\Lambda = \operatorname{span}_{\mathbb{Z}} S(\Lambda)$.
- It has been shown by Conway & Sloane (1995) and Martinet & Schürmann (2011) that there are lattices of rank ≥ 10 generated by minimal vectors which do not contain a **basis of minimal vectors**.

Lattices: frames of minimal vectors

If $\Lambda \subset \mathbb{R}^n$ is WR of full rank, then $S'(\Lambda)$, the set containing precisely one of each pair of $\pm$ vectors in $S(\Lambda)$ is a frame:

$$k = |S'(\Lambda)| = \frac{|S(\Lambda)|}{2} \geq n \text{ and } \operatorname{span}_{\mathbb{R}} S'(\Lambda) = \mathbb{R}^n.$$

Lattices: frames of minimal vectors

If $\Lambda \subset \mathbb{R}^n$ is WR of full rank, then $S'(\Lambda)$, the set containing precisely one of each pair of $\pm$ vectors in $S(\Lambda)$ is a frame:

$$k = |S'(\Lambda)| = \frac{|S(\Lambda)|}{2} \geq n \text{ and } \operatorname{span}_{\mathbb{R}} S'(\Lambda) = \mathbb{R}^n.$$

Coherence of such a frame, which we call **coherence of the lattice** Λ, is at most $1/2$, i.e.

$$C(\Lambda) := \max_{\boldsymbol{x}_i, \boldsymbol{x}_j \in S'(\Lambda)} \frac{|\langle \boldsymbol{x}_i, \boldsymbol{x}_j \rangle|}{|\Lambda|^2} \leq \frac{1}{2},$$

and for instance $C(\mathbb{Z}^n) = 0$.

Lattices: frames of minimal vectors

If $\Lambda \subset \mathbb{R}^n$ is WR of full rank, then $S'(\Lambda)$, the set containing precisely one of each pair of $\pm$ vectors in $S(\Lambda)$ is a frame:

$$k = |S'(\Lambda)| = \frac{|S(\Lambda)|}{2} \geq n \text{ and } \operatorname{span}_{\mathbb{R}} S'(\Lambda) = \mathbb{R}^n.$$

Coherence of such a frame, which we call **coherence of the lattice** Λ, is at most $1/2$, i.e.

$$C(\Lambda) := \max_{\boldsymbol{x}_i, \boldsymbol{x}_j \in S'(\Lambda)} \frac{|\langle \boldsymbol{x}_i, \boldsymbol{x}_j \rangle|}{|\Lambda|^2} \leq \frac{1}{2},$$

and for instance $C(\mathbb{Z}^n) = 0$.

If $|S'(\Lambda)| > n$, how small can $C(\Lambda)$ be?

Packing density

The **packing density** of a lattice Λ of rank n is defined as

$$\delta(\Lambda) = \frac{\omega_n |\Lambda|^n}{2^n \det \Lambda},$$

where ω_n is the volume of a unit ball in $\mathbb{R}^n$.

Packing density

The **packing density** of a lattice Λ of rank n is defined as

$$\delta(\Lambda) = \frac{\omega_n |\Lambda|^n}{2^n \det \Lambda},$$

where ω_n is the volume of a unit ball in $\mathbb{R}^n$.

Space of full-rank lattices in $\mathbb{R}^n$ is identified with $\mathrm{GL}_n(\mathbb{R})/\mathrm{GL}_n(\mathbb{Z})$, and δ is a continuous function on this space.

Packing density

The **packing density** of a lattice Λ of rank n is defined as

$$\delta(\Lambda) = \frac{\omega_n |\Lambda|^n}{2^n \det \Lambda},$$

where ω_n is the volume of a unit ball in $\mathbb{R}^n$.

Space of full-rank lattices in $\mathbb{R}^n$ is identified with $\operatorname{GL}_n(\mathbb{R}) / \operatorname{GL}_n(\mathbb{Z})$, and δ is a continuous function on this space.

Is there any connection between $C(\Lambda)$ and $\delta(\Lambda)$?

Lattices: eutaxy and perfection

Let $n = \operatorname{rk} \Lambda$ and

$$S(\Lambda) = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_{2k}\}$$

be the set of minimal vectors of the lattice Λ.

Lattices: eutaxy and perfection

Let $n = \operatorname{rk} \Lambda$ and

$$S(\Lambda) = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_{2k}\}$$

be the set of minimal vectors of the lattice Λ.

This lattice is called **eutactic** if there exist positive real numbers $c_1, \dots, c_{2k}$ such that

$$\|\boldsymbol{v}\|^2 = \sum_{i=1}^{2k} c_i \langle \boldsymbol{v}, \boldsymbol{x}_i \rangle^2$$

for every vector $\boldsymbol{v} \in \operatorname{span}_{\mathbb{R}} \Lambda$. If $c_1 = \dots = c_{2k}$, we say that Λ is **strongly eutactic**.

Lattices: eutaxy and perfection

Let $n = \operatorname{rk} \Lambda$ and

$$S(\Lambda) = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_{2k}\}$$

be the set of minimal vectors of the lattice Λ.

This lattice is called **eutactic** if there exist positive real numbers $c_1, \dots, c_{2k}$ such that

$$\|\boldsymbol{v}\|^2 = \sum_{i=1}^{2k} c_i \langle \boldsymbol{v}, \boldsymbol{x}_i \rangle^2$$

for every vector $\boldsymbol{v} \in \operatorname{span}_{\mathbb{R}} \Lambda$. If $c_1 = \dots = c_{2k}$, we say that Λ is **strongly eutactic**.

This lattice is called **perfect** if the set of symmetric matrices

$$\{\boldsymbol{x}_i \boldsymbol{x}_i^\top : \boldsymbol{x}_i \in S(\Lambda)\}$$

spans the space of $n \times n$ symmetric matrices.

Theorems of Voronoi and Ash

A lattice is called **extremal** if it is a local maximum of the packing density function in its dimension.

Theorems of Voronoi and Ash

A lattice is called **extremal** if it is a local maximum of the packing density function in its dimension.

Theorem 1 (G. Voronoi – 1908)

A lattice is extremal if and only if it is perfect and eutactic.

Theorems of Voronoi and Ash

A lattice is called **extremal** if it is a local maximum of the packing density function in its dimension.

Theorem 1 (G. Voronoi – 1908)

A lattice is extremal if and only if it is perfect and eutactic.

Theorem 2 (A. Ash – 1977)

Eutactic lattices are critical points of the packing density function.

Theorems of Voronoi and Ash

A lattice is called **extremal** if it is a local maximum of the packing density function in its dimension.

Theorem 1 (G. Voronoi – 1908)

A lattice is extremal if and only if it is perfect and eutactic.

Theorem 2 (A. Ash – 1977)

Eutactic lattices are critical points of the packing density function.

Both, perfect and eutactic lattices are WR. While it is possible to reduce the packing density of a eutactic lattice by locally modifying it to be non-WR, this is less clear if we stay in the WR space.

Theorems of Voronoi and Ash

A lattice is called **extremal** if it is a local maximum of the packing density function in its dimension.

Theorem 1 (G. Voronoi – 1908)

A lattice is extremal if and only if it is perfect and eutactic.

Theorem 2 (A. Ash – 1977)

Eutactic lattices are critical points of the packing density function.

Both, perfect and eutactic lattices are WR. While it is possible to reduce the packing density of a eutactic lattice by locally modifying it to be non-WR, this is less clear if we stay in the WR space.

Are non-perfect eutactic lattices necessarily the minima of the packing density function when restricted to the space of WR lattices?

Nearly orthogonal lattices: definitions

Let $B = \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_n\}$ be an ordered basis for a lattice L in $\mathbb{R}^n$, and define a sequence of angles $\theta_1, \dots, \theta_{n-1}$ as follows: each θ_i is the angle between $\boldsymbol{b}_{i+1}$ and the subspace $\text{span}_{\mathbb{R}}\{\boldsymbol{b}_1, \dots, \boldsymbol{b}_i\}$. It is then clear that each $\theta_i \in [0, \pi/2]$.

Nearly orthogonal lattices: definitions

Let $B = \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_n\}$ be an ordered basis for a lattice L in $\mathbb{R}^n$, and define a sequence of angles $\theta_1, \dots, \theta_{n-1}$ as follows: each θ_i is the angle between $\boldsymbol{b}_{i+1}$ and the subspace $\operatorname{span}_{\mathbb{R}}\{\boldsymbol{b}_1, \dots, \boldsymbol{b}_i\}$. It is then clear that each $\theta_i \in [0, \pi/2]$.

Basis B and lattice L are called **weakly nearly orthogonal** if $\theta_i \geq \pi/3$ for each $1 \leq i \leq n-1$; B and L are called **nearly orthogonal** if every ordering of it is weakly nearly orthogonal.

Nearly orthogonal lattices: definitions

Let $B = \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_n\}$ be an ordered basis for a lattice L in $\mathbb{R}^n$, and define a sequence of angles $\theta_1, \dots, \theta_{n-1}$ as follows: each θ_i is the angle between $\boldsymbol{b}_{i+1}$ and the subspace $\operatorname{span}_{\mathbb{R}}\{\boldsymbol{b}_1, \dots, \boldsymbol{b}_i\}$. It is then clear that each $\theta_i \in [0, \pi/2]$.

Basis B and lattice L are called **weakly nearly orthogonal** if $\theta_i \geq \pi/3$ for each $1 \leq i \leq n-1$; B and L are called **nearly orthogonal** if every ordering of it is weakly nearly orthogonal.

Theorem 3 (R. Neelamani, S. Dash, R. Baraniuk – 2007)

A weakly nearly orthogonal basis contains a minimal vector of L.

Nearly orthogonal lattices: definitions

Let $B = \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_n\}$ be an ordered basis for a lattice L in $\mathbb{R}^n$, and define a sequence of angles $\theta_1, \dots, \theta_{n-1}$ as follows: each θ_i is the angle between $\boldsymbol{b}_{i+1}$ and the subspace $\text{span}_{\mathbb{R}}\{\boldsymbol{b}_1, \dots, \boldsymbol{b}_i\}$. It is then clear that each $\theta_i \in [0, \pi/2]$.

Basis B and lattice L are called **weakly nearly orthogonal** if $\theta_i \geq \pi/3$ for each $1 \leq i \leq n-1$; B and L are called **nearly orthogonal** if every ordering of it is weakly nearly orthogonal.

Theorem 3 (R. Neelamani, S. Dash, R. Baraniuk – 2007)

A weakly nearly orthogonal basis contains a minimal vector of L.

Nearly orthogonal lattices have applications in image processing, signal recovery, and related areas.

Nearly orthogonal lattices: packing density

We write $\mathcal{W}_n$ for the space of all WR weakly nearly orthogonal lattices, and $\mathcal{W}_n^*$ for the space of all WR nearly orthogonal lattices in $\mathbb{R}^n$.

Nearly orthogonal lattices: packing density

We write $\mathcal{W}_n$ for the space of all WR weakly nearly orthogonal lattices, and $\mathcal{W}_n^*$ for the space of all WR nearly orthogonal lattices in $\mathbb{R}^n$.

We show that a lattice $L \in \mathcal{W}_n^*$ with coherence $0 < C(L) < 1/2$ can be, loosely speaking, **locally modified to increase or decrease the packing density by respectively increasing or decreasing $C(L)$**.

Nearly orthogonal lattices: packing density

We write $\mathcal{W}_n$ for the space of all WR weakly nearly orthogonal lattices, and $\mathcal{W}_n^*$ for the space of all WR nearly orthogonal lattices in $\mathbb{R}^n$.

We show that a lattice $L \in \mathcal{W}_n^*$ with coherence $0 < C(L) < 1/2$ can be, loosely speaking, **locally modified to increase or decrease the packing density by respectively increasing or decreasing $C(L)$**. In particular, we prove the following.

Theorem 4 (L. F., D. Kogan – 2021)

For any $n \geq 3$ the space $\mathcal{W}_n^*$ does not contain any perfect (and, hence, any extremal) lattices. On the other hand, $\mathcal{W}_n^*$ contains multiple eutactic and strongly eutactic lattices for every $n \geq 2$.

Nearly orthogonal lattices: packing density

Our argument explicitly shows how to locally modify a given lattice L staying within $\mathcal{W}_n^*$ to increase or decrease the packing density.

Nearly orthogonal lattices: packing density

Our argument explicitly shows how to locally modify a given lattice L staying within $\mathcal{W}_n^*$ to increase or decrease the packing density.

The packing density function δ has only one minimum on $\mathcal{W}_n^*$: the integer lattice $\mathbb{Z}^n$, which is a strongly eutactic lattice. On the other hand, we show constructions and examples of other eutactic and strongly eutactic lattices in $\mathcal{W}_n^*$, including irreducible ones (those that can be represented as orthogonal direct sums of sublattices).

Nearly orthogonal lattices: packing density

Our argument explicitly shows how to locally modify a given lattice L staying within $\mathcal{W}_n^*$ to increase or decrease the packing density.

The packing density function δ has only one minimum on $\mathcal{W}_n^*$: the integer lattice $\mathbb{Z}^n$, which is a strongly eutactic lattice. On the other hand, we show constructions and examples of other eutactic and strongly eutactic lattices in $\mathcal{W}_n^*$, including irreducible ones (those that can be represented as orthogonal direct sums of sublattices).

This shows that **non-perfect eutactic lattices are not necessarily the minima of the packing density function, even when restricted to the space of WR lattices**.

Nearly orthogonal lattices: minimal vectors

We also obtain bounds on the numbers of minimal vectors.

Theorem 5 (L. F., D. Kogan – 2021)

Let $L \in \mathcal{W}_n$, *then* $|S(L)| \leq 4n - 2$, *and every even number between* $2n$ *and* $4n - 2$ *is possible. Further, if* $L \in \mathcal{W}_n^*$ *then* $|S(L)| < 3n$.

Nearly orthogonal lattices: minimal vectors

We also obtain bounds on the numbers of minimal vectors.

Theorem 5 (L. F., D. Kogan – 2021)

Let $L \in \mathcal{W}_n$, then $|S(L)| \leq 4n - 2$, and every even number between $2n$ and $4n - 2$ is possible. Further, if $L \in \mathcal{W}_n^*$ then $|S(L)| < 3n$.

It is clear from the above discussion that WR nearly orthogonal lattices are not *just* WR: they have bases of minimal vectors.

Nearly orthogonal lattices: minimal vectors

We also obtain bounds on the numbers of minimal vectors.

Theorem 5 (L. F., D. Kogan – 2021)

Let $L \in \mathcal{W}_n$, then $|S(L)| \leq 4n-2$, and every even number between $2n$ and $4n-2$ is possible. Further, if $L \in \mathcal{W}_n^*$ then $|S(L)| < 3n$.

It is clear from the above discussion that WR nearly orthogonal lattices are not *just* WR: they have bases of minimal vectors. In fact, we can prove more.

Theorem 6 (L. F., D. Kogan – 2021)

Let $L \in \mathcal{W}_n^*$, then any n linearly independent vectors in $S(L)$ form a basis for L.

Nearly orthogonal lattices: coherence

We also discuss coherence of lattices in $\mathcal{W}_n^*$ in more details. Define

$$c_n = \frac{\sqrt{(n-2)^2 + 16(n-1)} - (n-2)}{8(n-1)}.$$

Nearly orthogonal lattices: coherence

We also discuss coherence of lattices in $\mathcal{W}_n^*$ in more details. Define

$$c_n = \frac{\sqrt{(n-2)^2 + 16(n-1)} - (n-2)}{8(n-1)}.$$

Theorem 7 (L. F., D. Kogan – 2021)

1. *For a lattice $L \in \mathcal{W}_n^*$, $C(L) = 1/2$ if and only if $|S(L)| > 2n$.*
2. *Let $B = \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_n\}$ in $\mathbb{R}^n$ be a collection of linearly independent unit vectors such that*

$$\max_{1 \leq i < j \leq n} |\langle \boldsymbol{b}_i, \boldsymbol{b}_j \rangle| \leq c_n.$$

Then $L = \operatorname{span}_{\mathbb{Z}} B$ is in $\mathcal{W}_n^*$.

Nearly orthogonal lattices: coherence

There is a famous family of lattices $A_n^* \subset \mathbb{R}^n$ for each $n \geq 2$ – this is the dual of the root lattice A_n. It is well known that A_n^* has $2(n+1)$ minimal vectors, so

$$|S'(A_n^*)| = n+1, \text{ and } C(A_n^*) = \frac{1}{n},$$

which gives equality in Welch lower bound on coherence.

Nearly orthogonal lattices: coherence

There is a famous family of lattices $A_n^* \subset \mathbb{R}^n$ for each $n \geq 2$ – this is the dual of the root lattice A_n. It is well known that A_n^* has $2(n+1)$ minimal vectors, so

$$|S'(A_n^*)| = n+1, \text{ and } C(A_n^*) = \frac{1}{n},$$

which gives equality in Welch lower bound on coherence.

In fact, $S'(A_n^*)$ is an equiangular frame, as expected. Hence **if $|S'(\Lambda)| > n$, $C(\Lambda)$ can be as small as $1/n$, but no smaller**.

Nearly orthogonal lattices: coherence

There is a famous family of lattices $A_n^* \subset \mathbb{R}^n$ for each $n \geq 2$ – this is the dual of the root lattice A_n. It is well known that A_n^* has $2(n+1)$ minimal vectors, so

$$|S'(A_n^*)| = n+1, \text{ and } C(A_n^*) = \frac{1}{n},$$

which gives equality in Welch lower bound on coherence.

In fact, $S'(A_n^*)$ is an equiangular frame, as expected. Hence **if $|S'(\Lambda)| > n$, $C(\Lambda)$ can be as small as $1/n$, but no smaller**.

Notice that $c_n < 1/n$, and $A_n^* \notin \mathcal{W}_n^*$, however

$$\lim_{n \to \infty} (c_n/(1/n)) = 1.$$

Hence lattices in $\mathcal{W}_n^*$ with $2n$ minimal vectors come asymptotically very close to this optimal family A_n^* with $2(n+1)$ minimal vectors.

Generic well-rounded lattices

A WR lattice $\Lambda \subset \mathbb{R}^n$ is called **generic** (abbreviated GWR) if it has precisely $2n$ minimal vectors. For example, $\mathbb{Z}^n$ is a GWR lattice.

Generic well-rounded lattices

A WR lattice $\Lambda \subset \mathbb{R}^n$ is called **generic** (abbreviated GWR) if it has precisely $2n$ minimal vectors. For example, $\mathbb{Z}^n$ is a GWR lattice.

The number of minimal vectors of a lattice is often called its **kissing number**: it is related to the kissing number problem in discrete geometry. Usually, people look for lattices with high kissing number.

Generic well-rounded lattices

A WR lattice $\Lambda \subset \mathbb{R}^n$ is called **generic** (abbreviated GWR) if it has precisely $2n$ minimal vectors. For example, $\mathbb{Z}^n$ is a GWR lattice.

The number of minimal vectors of a lattice is often called its **kissing number**: it is related to the kissing number problem in discrete geometry. Usually, people look for lattices with high kissing number.

For some coding models in wireless information theory, it is of interest to increase packing density while minimizing kissing number. GWR lattices come useful here.

Generic well-rounded lattices

A WR lattice $\Lambda \subset \mathbb{R}^n$ is called **generic** (abbreviated GWR) if it has precisely $2n$ minimal vectors. For example, $\mathbb{Z}^n$ is a GWR lattice.

The number of minimal vectors of a lattice is often called its **kissing number**: it is related to the kissing number problem in discrete geometry. Usually, people look for lattices with high kissing number.

For some coding models in wireless information theory, it is of interest to increase packing density while minimizing kissing number. GWR lattices come useful here.

Lattices in $\mathcal{W}_n^*$ with coherence $< 1/2$ (and hence kissing number $2n$) may be good candidates, since it may be possible to sufficiently increase their density by appropriate local modifications. This can be a project for the future.

Reference

L. Fukshansky and D. Kogan, *On the geometry of nearly orthogonal lattices*, Linear Algebra and its Applications, vol. 629 (2021), pg. 112–137

This paper is available at:

`http://math.cmc.edu/lenny/research.html`

Reference

L. Fukshansky and D. Kogan, *On the geometry of nearly orthogonal lattices*, Linear Algebra and its Applications, vol. 629 (2021), pg. 112–137

This paper is available at:

`http://math.cmc.edu/lenny/research.html`

Thank you!