LATTICE POINTS IN HOMOGENEOUSLY EXPANDING COMPACT DOMAINS

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Abstract. We state two problems in the general direction of counting lattice points in a compact domain.

A variety of interesting and important questions in geometric combinatorics and in geometry of numbers is connected to counting integer lattice points in compact subsets of a Euclidean space. In case such a subset is a rational polyhedron, the problem can be reformulated in terms of Ehrhart polynomial. In a general situation, however, one often has to rely on estimates of asymptotic nature. A good example of such an estimate is presented by S. Lang in Theorem 2 on p. 128 of [3]. We state it here. In the rest of this note we assume that $N > 1$ is an integer.

Theorem #1, ([3]). *Let D be a compact subset of $\mathbb{R}^N$, and let L be a lattice of rank N in $\mathbb{R}^N$ with fundamental domain F. Assume that the boundary ∂D of D is Lipschitz-parametrizable. Then for each positive $t \in \mathbb{R}$ the number of points of L in tD is given by the following asymptotic formula:*

$$|L \cap tD| = \frac{\operatorname{Vol}(D)}{\operatorname{Vol}(F)} t^N + O(t^{N-1}), \tag{1}$$

where Vol *stands for volume in $\mathbb{R}^N$, and the constant in O depends on L, N, and Lipschitz constants.*

We recall that the condition that ∂D is *Lipschitz-parametrizable* means that there exists a finite collection of maps $\varphi_j : [0,1]^N \longrightarrow \partial D$, the union of images of which covers ∂D and there exists a constant K such that for all $\boldsymbol{x}, \boldsymbol{z} \in [0,1]^N$

$$|\varphi_j(\boldsymbol{x}) - \varphi_j(\boldsymbol{z})| \leq K|\boldsymbol{x} - \boldsymbol{z}|,$$

for each j, where $|\ |$ stands for the sup-norm on $\mathbb{R}^N$, i.e. $|\boldsymbol{x}| = \max_{1 \leq i \leq N} |x_i|$. The constant K is called the associated *Lipschitz constant*.

Notice that the main term in the upper bound in (1) is explicit and easily computable, but the error term is implicit. Loosely speaking, the main term of such an asymptotic estimate counts the number of "interior points" of L in D, i.e. points that are away from the boundary, and the error term accounts for the points near the boundary. For practical applications it is important to be able to explicitly estimate the error term. Such an estimate was carried out for instance by H. Davenport in [1] (see also [5] for a very nice account and generalizations of Davenport's theorem). However Davenport's bound on the error term depends on projection volumes of D onto certain subspaces of $\mathbb{R}^N$ as well as determinants of projections of L onto these subspaces. These are hard to compute. In some situations one would prefer perhaps cruder, but more tractable bounds on the error term. An alternative approach to this problem is to try to "quantify" the original argument in Lang's theorem. This has been partially done by P. G. Spain in [4]. We briefly outline this approach here and ask some further questions.

We start by sketching out the main idea of proof of Theorem #1. One proceeds by noticing that for each positive real number t

$$m(t) \leq |L \cap tD| \leq m(t) + b(t),$$

where

1. $m(t) =$ number of $\boldsymbol{x} \in L$ such that $F + \boldsymbol{x} \subseteq$ interior of of tD,
2. $b(t) =$ number of $\boldsymbol{x} \in L$ such that $F + \boldsymbol{x}$ intersects $\partial(tD)$.

It is obvious that

$$m(t) \leq \frac{\mathrm{Vol}(tD)}{\mathrm{Vol}(F)} = \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(F)} t^N,$$

which produces the main term. In order to produce the error term one needs to estimate $b(t)$. This unfortunately is not so easy. Lang only proves that $b(t) = O(t^{N-1})$ using the fact that the boundary $\partial(tD)$ of tD is Lipschitz-parametrizable, but does not exhibit any explicit upper bound. Although, as we discussed above, there are other methods for estimating the error term, the quantity $b(t)$ seems to be interesting in its own right. It can, for instance, be related to a covering problem, namely: how many translates of the closure of the fundamental domain F does it take to cover the the compact domain tD? Such a number can again be approximated by the expression $m(t) + b(t)$ as above. The following estimate for $b(t)$ in the special case when $L = \mathbb{Z}^N$ was produced by P G. Spain.

Theorem #2, ([4]). *Let D be as in Theorem #1, so that the boundary ∂D is Lipschitz-parametrizable with Lipschitz constant K. Let $L = \mathbb{Z}^N$. Let F be the fundamental domain of L with respect to the standard basis, and let $b(t)$ as above be the number of translates of F that have nonempty intersection with $\partial(tD)$ where $t \geq 1/K$. Then*

$$b(t) \leq 2^N (Kt + 1)^{N-1}, \tag{2}$$

and therefore

$$|\mathbb{Z}^N \cap tD| \leq \mathrm{Vol}(D) t^N + 2^N (Kt+1)^{N-1} = \mathrm{Vol}(D) t^N + 2^N \sum_{i=0}^{N-1} K^i t^i. \tag{3}$$

Problem #1. *Provide an explicit bound on $b(t)$ as in Theorem #2 for a general lattice L.*

Perhaps one can modify Spain's argument to produce a solution to Problem #1. Notice that the upper bound on $|L \cap tD|$ as it comes out in (3) is a polynomial in t whose coefficients depend on volume of D and on the Lipschitz constant K. This suggests a certain analogy with Ehrhart polynomial: one may look for polynomial upper and lower bounds on $|L \cap tD|$ for some more or less general instances of L and D. Here is an example for a simple choice of D when L is any sublattice of $\mathbb{Z}^N$ of full rank. Let

$$C_t^N = \{\boldsymbol{y} \in \mathbb{R}^N : \max\{|y_1|, ..., |y_N|\} \leq t\},$$

that is C_t^N is a cube with side length $2t$ centered at the origin in $\mathbb{R}^N$.

Theorem #3, ([2]). *Let $\Lambda \subseteq \mathbb{Z}^N$ be a lattice of full rank in $\mathbb{R}^N$ of determinant Δ. Then for each point $\boldsymbol{z}$ in $\mathbb{R}^N$ we have*

$$\left(\frac{2^N}{\Delta}\right) t^N \leq |\Lambda \cap (C_t^N + \boldsymbol{z})| \leq \left(\frac{2t}{\Delta} + 1\right)(2t+1)^{N-1} = \left(\frac{2^N}{\Delta}\right) t^N + \sum_{i=1}^{N-1} 2^i \left(\frac{1}{\Delta} + 1\right) t^i + 1. \tag{4}$$

In [2] an analogous bound for a rectangular box instead of a cube is also produced; the result of Theorem #3 is extended to lattices of not full rank and to certain modules over the ring of algebraic integers in a number field, viewed as $\mathbb{Z}$-modules.

Problem #2. *Produce explicit polynomial bounds like (4) for more general choices of compact domain D.*

References

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