POINTS AND SUBSPACES OF SMALL HEIGHT IN QUADRATIC AND LINEAR SPACES

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Abstract. We investigate a few related questions on search bounds via height for rational points over global fields in quadratic and linear spaces. Specifically, let K be a global field or $\overline{\mathbb{Q}}$ and $N \geq 2$ an integer. Let $V \subseteq K^N$ be a vector space and F a quadratic form in N variables over K which is isotropic on V. We prove the existence of small-height totally isotropic subspaces of the quadratic space (V, F) in case K is a function field, which extends similar previous results of Vaaler in case K is a number field and the second author in case $K = \overline{\mathbb{Q}}$. Furthermore, we establish the existence of an infinite collection of generating families of maximal totally isotropic subspaces of bounded height for a quadratic space (V, F) over any global field K, generalizing a previous result of Vaaler. If in addition Z is a finite union of varieties over K such that $V \setminus Z$ contains a non-trivial zero of F, we prove the existence of a collection of small-height linearly independent zeros of F in V over K outside of Z. As a corollary of this result, we show that in fact there exist small-height maximal totally isotropic subspaces of the quadratic space (V, F) outside of Z. The paper also contains two appendices: the first one establishing the existence of a small-height basis for V over K outside of Z, and the second containing an effective version of Cartan-Dieudonné theorem over a function field; these are extensions of previous result of the second author. Our results contribute to the effective study of quadratic and linear forms via height in the general spirit of Siegel's lemma and Cassels' theorem on small zeros of quadratic forms. All bounds on height are explicit.

Contents

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1. Introduction and statement of results

The questions of existence of rational points on varieties over global fields and rings are usually quite difficult and in some cases are known to be undecidable, as confirmed by Matiyasevich's famous negative answer [16] to Hilbert's tenth problem. Moreover, it has been shown by J. P. Jones [14] that the question whether a general Diophantine equation of degree four or larger has a solution in positive integers is already undecidable. This serves as a certain indication that a general finite search algorithm for rational points on projective hypersurfaces of degree larger than three may be quite difficult to obtain, if possible at all. In case of cubic forms, the available literature is quite limited with only some partial results over the rational integers. The linear and quadratic cases however have been extensively studied over a variety of global fields.

The approach we will concentrate on involves *height functions*, a standard tool of arithmetic geometry, the choice of which we specify in Section 2 below. A key property of height over a fixed number field is that the set of all projective points with height $\leq C$ is finite for every choice of the positive real constant C. Suppose now that there exists a point on a given variety V over a fixed number field K, and say that we can prove that under this assumption there must exist a point of height bounded above by some explicit constant C. This means that in principle there exists a finite search algorithm which allows one to determine whether V has any points rational over K, and hence we will call C a *search bound* with respect to height.

Search bounds for hypersurfaces of degree ≥ 4 currently seem to be out of reach, and still relatively little is known about the cubic hypersurfaces; see [2] for the most recent results and a survey of the previous work. Hence the investigation of search bounds in the linear and quadratic cases (including situations with additional algebraic conditions imposed) helps to explore the boundaries of Diophantine undecidability, in addition to being of intrinsic arithmetic interest. In this paper we present a selection of results on search bounds for linear and quadratic spaces. The simplest problem along these lines asks for a bound on height of a nonzero point in vector space over a global field. A more general result is given by a modern version of Siegel's lemma, which guarantees the existence of a full small-height basis for a vector space (proved over number fields by Bombieri and Vaaler [1], over function fields by Thunder [20], and over their algebraic closures by Roy and Thunder [17]). The Faltings version of Siegel's lemma in its original form (see [7], [15], [6]) asserts the existence of a point of small height in a vector space over $\mathbb{R}$ outside of a proper subspace. A much more general version of this principle over global fields, asserting the existence of such a point in a vector space outside of a finite union of varieties, has been established in [11]. Furthermore, a generalization of the main result of [11] to a statement on the existence of a full small-height basis satisfying the same algebraic conditions is presented as Theorem A.1 in Appendix A at the end of this paper.

In the present paper we will primarily concentrate on the effective study of quadratic spaces. The investigation of small-height zeros of quadratic forms was initiated in the celebrated paper of Cassels [4], and later continued by a number of authors (see [5] for an overview). In particular, in [21] Vaaler proved the existence of a small-height maximal totally isotropic subspace of an L-dimensional quadratic space over a fixed number field, and in [22] established the existence of a whole family of L such subspaces generating the entire space. In [10] an analogue of Vaaler's result [21] was established over $\overline{\mathbb{Q}}$. Our first result establishes an analogue of Vaaler's result over function fields with finite constant fields. All the necessary notation is introduced in Section 2.

Theorem 1.1. *Let K be a function field over a finite field $\mathbb{F}_q$ for some odd prime power q, and let F be a nonzero quadratic form in $K[X_1, \dots, X_N]$. Let $V \subseteq K^N$ be an L-dimensional vector space, $1 \leq L \leq N$. Suppose that the quadratic space (V, F) has a totally isotropic subspace of dimension $\ell \geq 1$, where ℓ is greater than the dimension of the radical of (V, F). Then there exists a totally isotropic subspace $\mathcal{A} \subseteq V$ of dimension ℓ such that*

$$H(\mathcal{A}) \leq q^{(L-\ell)^2 g(K)/d} H(F)^{(L-\ell)/2} H(V). \tag{1}$$

We prove Theorem 1.1 (along with some useful corollaries) in Section 3, and then derive from it a function field version of effective Witt decomposition (Corollary 3.11) using the same type of argument as in [9] and [10].

Our next theorem is a generalization and extension over $\overline{\mathbb{Q}}$ of Vaaler's result [22], establishing the existence of infinitely many such generating families of maximal totally isotropic subspaces of bounded height.

Theorem 1.2. *Let K be a number field (abbreviated n.f.), a function field over a finite field $\mathbb{F}_q$ for some odd prime power q (abbreviated f.f.f.), or $\overline{\mathbb{Q}}$, and let $V \subseteq K^N$ be an L-dimensional vector space, $1 \leq L \leq N$. Let F be a quadratic form in N variables defined over K, let ω be the Witt index of the quadratic space (V, F), let λ be the dimension of its radical $V^{\perp}$, and let $r := L - \lambda$ be the rank of F on V. Define $k := \left[\frac{r}{2}\right]$ and $J := r - 2\omega$. Then there exists an infinite family of $(\omega+\lambda)$-dimensional subspaces of V such that F vanishes identically on each of them.*

Specifically, if $J > 0$, then this family $\left\{W^n_{ij}\right\}$ is indexed by triples of indices (n, i, j) where $n \geq 1$, $1 \leq i \leq \omega$, $1 \leq j \leq J$. Moreover, for each $n \geq 1$,

$$\dim_K W^n_{ij} \cap W^n_{i'j'} = \omega + \lambda - 3,$$

and

$$\operatorname{span}_K \{W^n_{ij} : 1 \leq i \leq \omega,\ 1 \leq j \leq J\} = V.$$

In addition, for each triple (n, i, j),

$$\mathcal{H}(W^n_{ij}) \leq \begin{cases} \mathfrak{C}_K\ a_K(n) H(F)^{\frac{(L+2\omega)\,\mathrm{p}_1(\omega)+4(\omega+r+9)}{8}} \mathcal{H}(V)^{\frac{\mathrm{p}_1(\omega)+2}{2}} & \text{if } K \text{ n.f. or f.f.f.} \\ \mathfrak{C}_K H(F)^{\frac{\omega+9+2r}{2}+\mathrm{p}_2(k,\omega)(k^2+1)} \mathcal{H}(V)^{\mathrm{p}_3(k,\omega)+2} & \text{if } K = \overline{\mathbb{Q}}, \end{cases} \tag{2}$$

where

$$\left.\begin{array}{l} \mathrm{p}_1(\omega) = (\omega+1)(\omega+2)(\omega+10) \\ \mathrm{p}_2(k,\omega) = (\omega+5)(k+1)(k+2)\left(\frac{3}{2}\right)^k \\ \mathrm{p}_3(k,\omega) = \frac{(6k+9)+\mathrm{p}_2(k,\omega)(6k+5)}{4k+2} \end{array}\right\}, \tag{3}$$

the constant $\mathfrak{C}_K = \mathfrak{C}_K(N, \omega, J, L, r)$ is defined by (75) and (76) below, and

$$a_K(n) = \begin{cases} n^2 & \text{if } K \text{ n.f.} \\ e^{2n} & \text{if } K \text{ f.f.f.} \\ 1 & \text{if } K = \overline{\mathbb{Q}}. \end{cases} \tag{4}$$

On the other hand, if $J=0$, then this family $\left\{W_{ij}^n\right\}$ is indexed by triples of indices (n,i,j) where $n \geq 1$, $1 \leq i \neq j \leq \omega$. Moreover, for each $n \geq 1$,

$$\dim_K W_{ij}^n \cap W_{i'j'}^n = \omega+\lambda-2 \text{ or } \omega+\lambda-4,$$

and

$$\operatorname{span}_K \{W_{ij}^n : 1 \leq i \neq j \leq \omega\} = V.$$

In addition, for each triple (n,i,j),

(5)

$$\mathcal{H}(W_{ij}^n) \leq \begin{cases} \mathfrak{F}_K \; a_K(n) H(F)^{\frac{(L+2\omega)\,\mathrm{q}_1(\omega)}{4}+\frac{\omega+r}{2}+8} \mathcal{H}(V)^{\mathrm{q}_1(\omega)+1} & \text{if } K \text{ n.f. or f.f.f.} \\ \mathfrak{F}_K H(F)^{(k^2+1)\,\mathrm{q}_2(k,\omega)+\frac{\omega}{2}+r+7} \mathcal{H}(V)^{\frac{(6k+5)\,\mathrm{q}_2(k,\omega)}{4k+2}+2} & \text{if } K=\overline{\mathbb{Q}}, \end{cases}$$

where

$$\left.\begin{array}{l} \mathrm{q}_1(\omega) = \frac{(\omega+1)(\omega+2)(\omega+14)}{2} \\ \mathrm{q}_2(k,\omega) = (\omega+10)(k+1)(k+2)\left(\frac{3}{2}\right)^k \end{array}\right\}, \tag{6}$$

and the constant $\mathfrak{F}_K = \mathfrak{F}_K(N,\omega,L,r)$ is defined by (78) and (79) below.

The basic notation of the arithmetic theory of quadratic forms, which is used in the statement of Theorem 1.2, is reviewed below in Section 2, along with definitions of the appropriate height functions and field constants.

Remark 1.1. Recall that, as a set, the *Fano variety* of m-planes on a projective variety $\mathcal{X}_K$ defined over a field K, which we denote by $\mathcal{F}_m(\mathcal{X}_K)$, is the set of m-dimensional vector spaces over K that are contained in $\mathcal{X}_K$; this is a natural generalization of a Grassmannian. If we adopt the notation of Theorem 1.2 and let $\mathcal{X}_K(F)$ be the hypersurface defined by the quadratic form F over K, then Theorem 1.2 can be interpreted as a statement about the existence of infinite families of points of bounded height on $\mathcal{F}_{\omega+\lambda}(\mathcal{X}_K(F))$.

Our next result attempts to infuse the effective theory of quadratic forms with the elements of Theorem A.1 by proving the existence of a small-height collection of linearly independent vectors in a quadratic space lying outside of an arbitrary finite union of varieties, not containing the space. Let $J \geq 1$ be an integer. For each $1 \leq i \leq J$, let $k_i \geq 1$ be an integer and let

$$P_{i1}(X_1,\dots,X_N),\dots,P_{ik_i}(X_1,\dots,X_N)$$

be polynomials of respective degrees $m_{i1},\dots,m_{ik_i} \geq 1$. Let

$$Z_K(P_{i1},\dots,P_{ik_i}) = \{\boldsymbol{x} \in K^N : P_{i1}(\boldsymbol{x}) = \dots = P_{ik_i}(\boldsymbol{x}) = 0\},$$

and define

$$\mathcal{Z}_K = \bigcup_{i=1}^J Z_K(P_{i1},\dots,P_{ik_i}). \tag{7}$$

For each $1 \leq i \leq J$ let $M_i = \max_{1 \leq j \leq k_i} m_{ij}$, and define

$$M = M(\mathcal{Z}_K) := \sum_{i=1}^J M_i. \tag{8}$$

Theorem 1.3. *Let K be a number field (n.f.), function field over a finite field $\mathbb{F}_q$ for some odd prime power q (f.f.f.), or $\overline{\mathbb{Q}}$, and let $V \subseteq K^N$ be an L-dimensional vector space, $1 \leq L \leq N$. Let F be a quadratic form in N variables defined over K. Let ω be the Witt index of the quadratic space (V,F), λ the dimension of its radical $V^\perp$, $r = L - \lambda$ the rank of F on V, and let $m = \omega + \lambda$ be the dimension of a maximal totally isotropic subspace of (V,F). Let*

$$Z(V,F) = \{\boldsymbol{z} \in V \setminus \{\boldsymbol{0}\} : F(\boldsymbol{z}) = 0\}.$$

Let $\mathcal{Z}_K$ and $M = M(\mathcal{Z}_K)$ be as in (7), (8) above. Suppose that $Z(V,F) \nsubseteq \mathcal{Z}_K$. Then there exist m linearly independent vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$ in V over K such that $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m \in Z(V,F) \setminus \mathcal{Z}_K$,

$$H(\boldsymbol{x}_1) \leq H(\boldsymbol{x}_2) \leq \dots \leq H(\boldsymbol{x}_m),\ \ h(\boldsymbol{x}_1) \leq h(\boldsymbol{x}_2) \leq \dots \leq h(\boldsymbol{x}_m), \tag{9}$$

and for each $1 \leq n \leq m$,

$$H(\boldsymbol{x}_n) \leq h(\boldsymbol{x}_n) \leq \begin{cases} T_K(L,M+1) H(F)^{\frac{9L+11}{2}} \mathcal{H}(V)^{9L+12} & \text{if } K \text{ n.f. or f.f.f.} \\ T_K(L,M+1) H(F)^{\max\{r,29/2\}} \mathcal{H}(V)^{30} & \text{if } K = \overline{\mathbb{Q}}, \end{cases} \tag{10}$$

where the constant $T_K(L,M,N)$ is defined by (90), (91), and (92) below.

As a corollary of Theorem 1.3, we can also obtain a statement on the existence of totally isotropic subspaces of (V,F) of bounded height missing the union of varieties $\mathcal{Z}_K$.

Corollary 1.4. *Let the notation be as in the statement of Theorem 1.3. Then for each pair of indices (n,k) with $1 \leq n, k \leq m$ there exists a totally isotropic subspace W_n^k of (V,F) such that $\dim_K W_n^k = k$,*

$$\operatorname{span}_K\{\boldsymbol{x}_n\} = W_n^1 \subset W_n^2 \subset \dots \subset W_n^m,$$

and so $W_n^k \nsubseteq \mathcal{Z}_K$ for each $1 \leq k \leq m$; also

$$\mathcal{H}(W_n^m) \leq \begin{cases} T_K^1(L,M,N,m) H(F)^{10L-m+11} \mathcal{H}(V)^{18L+25} & \text{if } K \text{ n.f. or f.f.f.} \\ T_K^2(L,M,N,\omega) H(F)^{\mathrm{t}_1(\omega,r)} \mathcal{H}(V)^{\mathrm{t}_2(\omega)} & \text{if } K = \overline{\mathbb{Q}}, \end{cases} \tag{11}$$

where the exponents in the $\overline{\mathbb{Q}}$ case are given by

$$\begin{aligned} \mathrm{t}_1(\omega,r) &= (\omega-1)^2 + \frac{4\omega+4+(4\omega+7)\max\{r,29/2\}}{3} + r, \\ \mathrm{t}_2(\omega) &= 40\omega + 70 + \frac{4\omega+4}{3}, \end{aligned} \tag{12}$$

and the constants are given by

$$T_K^1(L,M,N,m) = 2^{(2m+1)(L-m-1)} B_K(L-m-1)^{2(L-m-1)} N^2 T_K(L,M+1)^2, \tag{13}$$

when K is a number field,

$$T_K^1(L,M,N,m) = q^{\frac{(L-m-1)^2 g(K)}{d}} T_K(L,M+1)^2, \tag{14}$$

when K is a function field, and

$$T_K^2(L,M,N,\omega) = 3^{2(\omega-1)\omega^3 + \frac{(4\omega+1)(L-1)(L-2)}{6}} N^{4\omega+\frac{5}{2}} T_K(L,M+1)^{\frac{4\omega+7}{3}}, \tag{15}$$

when $K = \overline{\mathbb{Q}}$; notice that in this case $\omega = \left[\frac{L-\lambda}{2}\right]$. In addition, for each $1 \leq k < m$,

$$\mathcal{H}(W_n^k) \leq N^{\delta k/2} C_K(m) \mathcal{E}_K(m)^{1-\delta} H(\boldsymbol{x}_n) \mathcal{H}(W_n^m), \tag{16}$$

where the constants $C_K(m)$ *and* $\mathcal{E}_K(m)$ *are defined by* (19) *and* (18)*, respectively, and* δ *is as in* (28)*.*

In fact, Theorem 1.2 along with a finiteness property of certain Fano varieties allows us to prove the existence of infinite collections of maximal totally isotropic subspaces of bounded height outside of a hypersurface for quadratic forms of maximal Witt index. Given a vector space V over a field K, we write $\mathbb{P}(V)$ for the projective space over V.

Corollary 1.5. *Let* K *be a number field, function field over a finite field* $\mathbb{F}_q$ *for some odd prime power* q*, or* $\overline{\mathbb{Q}}$*, and let* F *be a quadratic form in* N *variables so that*

$$N = 2\omega + \lambda, \tag{17}$$

where ω *is the Witt index of* F *on* K^N *and* λ *is the dimension of the radical of* (K^N, F)*, as in Theorem 1.2. Let* $\left\{W^n_{ij}\right\}$ *be the family of maximal totally isotropic subspaces of* (K^N, F) *of bounded height as given by Theorem 1.2. Let* P *be a nonsingular homogeneous polynomial of degree* $d \geq 3$ *in* N *variables and write* $\mathcal{X}_K(P)$ *for the projective hypersurface it defines. Then for infinitely many triples* (n, i, j)*,* $\mathbb{P}(W^n_{ij}) \nsubseteq \mathcal{X}_K(P)$ *and* $\mathcal{H}(W^n_{ij})$ *is bounded as in* (5)*.*

Proof. A theorem of Debarre (see Corollary in the appendix of [3]) implies that when (17) is satisfied, $\mathcal{F}_{\omega+\lambda}(\mathcal{X}_K(P))$ is finite; in fact, its cardinality is bounded above by $O\left(d^{(\omega+\lambda+1)^2}\right)$. Moreover, if $\lambda > 0$ then Proposition A.1 in the appendix of [3] implies that $\mathcal{F}_{\omega+\lambda}(\mathcal{X}_K(P))$ is empty. Since $\left\{W^n_{ij}\right\}$ is an infinite family of $(\omega+\lambda)$-dimensional subspaces of K^N, the conclusion follows. □

Remark 1.2. In fact, there exists such W^n_{ij} with $n \leq O\left(d^{(\omega+\lambda+1)^2}\right)$. In other words, under the condition (17) there exist infinitely many points in $\mathcal{F}_{\omega+\lambda}(\mathcal{X}_K(F)) \setminus \mathcal{F}_{\omega+\lambda}(\mathcal{X}_K(P))$ whose height is bounded above by the expressions on the right hand side of (5). Notice in particular that the condition (17) is automatically satisfied by any nonsingular quadratic form in even number of variables over $\overline{\mathbb{Q}}$.

This paper is organized as follows. In Section 2 we set the notation, define the necessary constants and height functions, and review the basic terminology in the algebraic theory of quadratic forms. We then present the proofs of Theorem 1.1 in Section 3, Theorem 1.2 in Section 4, and Theorem 1.3 along with Corollary 1.4 in Section 5. Appendix A contains two variations of Siegel's lemma: one on small-height basis for a vector space over number field, function field, or $\overline{\mathbb{Q}}$ outside of a finitie union of varieties, generalizing the main result of [11] (Theorem A.1), and an "orthogonal" version of Siegel's lemma for a bilinear space over a function field (Theorem A.2), which is an analogue of Theorem 2.4 of [9] over a number field and Theorem 6.1 of [10] over $\overline{\mathbb{Q}}$. Finally, Appendix B contains related results on small-height isometries of quadratic spaces and effective version of Cartan-Dieudonné theorem over a function field; this is an analogue of Theorem 1.4 of [9] over a number field and Theorem 6.4 of [10] over $\overline{\mathbb{Q}}$.

2. Notation and heights

We start with some notation, following [11] and [9]. For the rest of this section, unless explicitly specified otherwise, we will assume that K is either a number field or a function field, and will write $\overline{K}$ for its algebraic closure; in particular, when K is a number field, $\overline{K} = \overline{\mathbb{Q}}$. By a function field we will always mean a finite separable algebraic extension of the field $\mathfrak{K} = \mathfrak{K}_0(t)$ of rational functions in one variable over a field $\mathfrak{K}_0$, where $\mathfrak{K}_0$ can be any perfect field. In the number field case, we write $d = [K : \mathbb{Q}]$ for the global degree of K over $\mathbb{Q}$; in the function field case, the global degree is $d = [K : \mathfrak{K}]$, and we also define the effective degree of K over $\mathfrak{K}$ to be

$$\mathfrak{m}(K, \mathfrak{K}) = \frac{[K : \mathfrak{K}]}{[K_0 : \mathfrak{K}_0]},$$

where K_0 is the algebraic closure of $\mathfrak{K}_0$ in K. If K is a number field, we let $\mathcal{D}_K$ be its discriminant, ω_K the number of roots of unity in K, r_1 its number of real embeddings, and r_2 its number of conjugate pairs of complex embeddings, so $d = r_1 + 2r_2$. If K is a function field, we will also write $g = g(K)$ for the genus of K, as defined by the Riemann-Roch theorem (see [20] for details). We will also need to define $\partial = \partial(K) := \min_{v \in M(K)} \deg(v)$, and let

$$\mathcal{E}_K(L) = e^{\frac{\partial g L}{d}}. \tag{18}$$

We will distinguish two cases: if K is a function field, we say that it is of *finite type* q if its subfield of constants is a finite field $\mathbb{F}_q$ for some prime power q, and we say that it is of *infinite type* if its subfield of constants is infinite. If K is of finite type, then $\partial = 1$.

Remark 2.1. Everywhere throughout this paper except for Appendices A and B we will be working with function fields of finite type only with q being an *odd* prime power - we will always mean this when referring to a function field of finite type.

If K is a function field of finite type q, then there exists a unique (up to isomorphism) smooth projective curve Y over $\mathbb{F}_q$ such that $K = \mathbb{F}_q(Y)$ is the field of rational functions on Y. In this case, we will write $n(K) = |Y(\mathbb{F}_q)|$ for the number of points of Y over $\mathbb{F}_q$, and h_K for the number of divisor classes of degree zero (which is precisely the cardinality of the Jacobian of Y over $\mathbb{F}_q$). We can now define the field constant $C_K(L)$:

(19)
$$C_K(L) = \begin{cases} \left(\left(\frac{2}{\pi}\right)^{r_2} |\mathcal{D}_K| \right)^{\frac{L}{2d}} & \text{if } K \text{ is a number field,} \\ \exp\left(\frac{(g(K)-1+\mathfrak{m}(K,\mathfrak{K}))L}{\mathfrak{m}(K,\mathfrak{K})} \right) & \text{if } K \text{ is a function field,} \\ e^{\frac{L(L-1)}{4}} + \varepsilon & \text{if } K = \overline{\mathbb{Q}}\text{; here we can take any } \varepsilon > 0, \end{cases}$$

and the constant $A_K(M)$:

(20)
$$A_K(M) = \begin{cases} \left(M\sqrt{2^{r_1}|\mathcal{D}_K|} \right)^{\frac{1}{d}} & \text{if } K \text{ is a number field with } \omega_K \leq M, \\ e^{R_K(M)} & \text{if } K \text{ is a function field of finite type } q \leq M, \\ 1 & \text{otherwise,} \end{cases}$$

for all integers $L, M \geq 1$, where for a function field K of finite type $q \leq M$ we define

$$R_K(M) = \frac{n(K)-1}{2} \left((M-q+2) h_K \sqrt{n(K)} \right)^{\frac{1}{n(K)-1}} + h_K (n(K)-1) \sqrt{n(K)}. \tag{21}$$

The constants $C_K(L)$ and $A_K(M)$ will appear in our various height estimates below.

Next we discuss absolute values on K. Let $M(K)$ be the set of all places of K when K is a number field or $\overline{\mathbb{Q}}$, and the set of all places of K which are trivial over the field of constants when K is a function field. For each place $v \in M(K)$ we write K_v for the completion of K at v and let d_v be the local degree of K at v, which is $[K_v : \mathbb{Q}_v]$ in the number field case, and $[K_v : \mathfrak{K}_v]$ in the function field case.

If K is a number field, then for each place $v \in M(K)$ we define the absolute value $|\ |_v$ to be the unique absolute value on K_v that extends either the usual absolute value on $\mathbb{R}$ or $\mathbb{C}$ if $v | \infty$, or the usual p-adic absolute value on $\mathbb{Q}_p$ if $v | p$, where p is a prime. For each $v | \infty$ and each positive integer j, define as in [21]

$$r_v(j) = \begin{cases} \pi^{-1/2} \Gamma(j/2+1)^{1/j} & \text{if } v|\infty \text{ is real,} \\ (2\pi)^{-1/2} \Gamma(j+1)^{1/2j} & \text{if } v|\infty \text{ is complex.} \end{cases}$$

We can now define the field constant

$$B_K(j) = 2 |\mathcal{D}_K|^{1/2d} \prod_{v|\infty} r_v(j)^{d_v/d}, \tag{22}$$

which appears in the statement of Theorem 1.2.

If K is a function field, then all absolute values on K are non-archimedean. For each $v \in M(K)$, let $\mathfrak{O}_v$ be the valuation ring of v in K_v and $\mathfrak{M}_v$ the unique maximal ideal in $\mathfrak{O}_v$. We choose the unique corresponding absolute value $|\ |_v$ such that:

(i) if $1/t \in \mathfrak{M}_v$, then $|t|_v = e$,
(ii) if an irreducible polynomial $p(t) \in \mathfrak{M}_v$, then $|p(t)|_v = e^{-\deg(p)}$.

In both cases, for each non-zero $a \in K$ the *product formula* reads

$$\prod_{v \in M(K)} |a|_v^{d_v} = 1. \tag{23}$$

We extend absolute values to vectors by defining the local heights. For each $v \in M(K)$ define a local height H_v on K_v^N by

$$H_v(\boldsymbol{x}) = \max_{1 \leq i \leq N} |x_i|_v^{d_v},$$

for each $\boldsymbol{x} \in K_v^N$. Also, for each $v | \infty$ we define another local height

$$\mathcal{H}_v(\boldsymbol{x}) = \left(\sum_{i=1}^N |x_i|_v^2 \right)^{d_v/2}.$$

Then we can define two slightly different global height functions on K^N:

$$H(\boldsymbol{x}) = \left(\prod_{v \in M(K)} H_v(\boldsymbol{x}) \right)^{1/d}, \quad \mathcal{H}(\boldsymbol{x}) = \left(\prod_{v \nmid \infty} H_v(\boldsymbol{x}) \times \prod_{v | \infty} \mathcal{H}_v(\boldsymbol{x}) \right)^{1/d}, \tag{24}$$

for each $\boldsymbol{x} \in K^N$. These height functions are *homogeneous*, in the sense that they are defined on the projective space $\mathbb{P}(K^N)$ thanks to the product formula (23):

$H(a\boldsymbol{x}) = H(\boldsymbol{x})$ and $\mathcal{H}(a\boldsymbol{x}) = \mathcal{H}(\boldsymbol{x})$ for any $\boldsymbol{x} \in K^N$ and $0 \neq a \in K$. It is easy to see that

$$H(\boldsymbol{x}) \leq \mathcal{H}(\boldsymbol{x}) \leq \sqrt{N} H(\boldsymbol{x}). \tag{25}$$

Notice that in case K is a function field, $M(K)$ contains no archimedean places, and so $H(\boldsymbol{x}) = \mathcal{H}(\boldsymbol{x})$ for all $\boldsymbol{x} \in K^N$. We also define the *inhomogeneous* height

$$h(\boldsymbol{x}) = H(1, \boldsymbol{x}),$$

which generalizes the Weil height on algebraic numbers: for each $\alpha \in K$, define

$$h(\alpha) = \prod_{v \in M(K)} \max\{1, |\alpha|_v\}^{d_v/d}.$$

Clearly, $h(\boldsymbol{x}) \geq H(\boldsymbol{x})$ for each $\boldsymbol{x} \in K^N$. All our inequalities will use heights H and h for vectors, however we use $\mathcal{H}$ to define the conventional Schmidt height on subspaces in the manner described below. This choice of heights coincides with [1].

We extend both heights H and $\mathcal{H}$ to polynomials by viewing them as height functions of the coefficient vector of a given polynomial. We also define a height function on subspaces of K^N. Let $V \subseteq K^N$ be a subspace of dimension L, $1 \leq L \leq N$. Choose a basis $\boldsymbol{x}_1, ..., \boldsymbol{x}_L$ for V, and write $X = (\boldsymbol{x}_1 \ ... \ \boldsymbol{x}_L)$ for the corresponding $N \times L$ basis matrix. Then

$$V = \{X\boldsymbol{t} : \boldsymbol{t} \in K^L\}.$$

On the other hand, there exists an $(N-L) \times N$ matrix A with entries in K such that

$$V = \{\boldsymbol{x} \in K^N : A\boldsymbol{x} = 0\}.$$

Let $\mathcal{I}$ be the collection of all subsets I of $\{1, ..., N\}$ of cardinality L. For each $I \in \mathcal{I}$ let I' be its complement, i.e. $I' = \{1, ..., N\} \setminus I$, and let $\mathcal{I}' = \{I' : I \in \mathcal{I}\}$. Then

$$|\mathcal{I}| = \binom{N}{L} = \binom{N}{N-L} = |\mathcal{I}'|.$$

For each $I \in \mathcal{I}$, write X_I for the $L \times L$ submatrix of X consisting of all those rows of X which are indexed by I, and ${}_{I'}A$ for the $(N-L) \times (N-L)$ submatrix of A consisting of all those columns of A which are indexed by I'. By the duality principle of Brill-Gordan [12] (also see Theorem 1 on p. 294 of [13]), there exists a non-zero constant $\gamma \in K$ such that

$$\det(X_I) = (-1)^{\varepsilon(I')} \gamma \det({}_{I'}A), \tag{26}$$

where $\varepsilon(I') = \sum_{i \in I'} i$. Define the vectors of *Grassmann coordinates* of X and A respectively to be

$$Gr(X) = (\det(X_I))_{I \in \mathcal{I}} \in K^{|I|}, \quad Gr(A) = (\det({}_{I'}A))_{I' \in \mathcal{I}'} \in K^{|I'|},$$

and so by (26) and (23)

$$\mathcal{H}(Gr(X)) = \mathcal{H}(Gr(A)).$$

Define the height of V, denoted by $\mathcal{H}(V)$, to be this common value. This definition is legitimate, since it does not depend on the choice of the basis for V. In particular, notice that if

$$\mathcal{L}(X_1, ..., X_N) = \sum_{i=1}^{N} q_i X_i \in K[X_1, ..., X_N]$$

is a linear form with a non-zero coefficient vector $\boldsymbol{q} \in K^N$, and $V = \{\boldsymbol{x} \in K^N : \mathcal{L}(\boldsymbol{x}) = 0\}$ is an $(N-1)$-dimensional subspace of K^N, then

$$\mathcal{H}(V) = \mathcal{H}(\mathcal{L}) = \mathcal{H}(\boldsymbol{q}). \tag{27}$$

Thus, given an $M \times N$ matrix A, we can talk about two different heights of A: the vector height $H(A)$, thinking of A as a vector in K^{MN}, and the subspace height $\mathcal{H}(A)$, thinking of A as a basis matrix of a vector space spanned over K by column vectors of A (or row vectors of A - depending on whether $M \leq N$ or $N \leq M$).

An important observation is that due to the normalizing exponent $1/d$ in (24) all our heights are *absolute*, meaning that they do not depend on the number field or function field of definition, hence are well defined over $\overline{K}$. In fact, throughout this paper the height of a vector over $\overline{K}$ is always computed over the smallest subfield of $\overline{K}$ containing this vector's coordinates. We should also remark that if K is a number field or a function field of finite type, then all our heights satisfy the *Northcott finiteness property*: for any positive real number C, the sets

$$\left\{[\boldsymbol{x}] \in \mathbb{P}(K^N) : H(\boldsymbol{x}) \leq C\right\},\ \left\{[\boldsymbol{x}] \in \mathbb{P}(K^N) : \mathcal{H}(\boldsymbol{x}) \leq C\right\},\ \left\{\boldsymbol{x} \in K^N : h(\boldsymbol{x}) \leq C\right\}$$

are finite; here $[\boldsymbol{x}]$ stands for the projective point represented by $\boldsymbol{x} \in K^N$. It is easy to see that in the cases when $K = \overline{K}$ or K is a function field of infinite type, this finiteness property no longer holds.

Let

$$\delta = \begin{cases} 1 & \text{if } K \text{ is a number field or } \overline{\mathbb{Q}} \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

We will also need a few technical lemmas detailing some basic properties of heights. The first one bounds the height of a linear combination of vectors.

Lemma 2.1. *For $\xi_1, ..., \xi_L \in K$ and $\boldsymbol{x}_1, ..., \boldsymbol{x}_L \in K^N$,*

$$H\left(\sum_{i=1}^{L} \xi_i \boldsymbol{x}_i\right) \leq h\left(\sum_{i=1}^{L} \xi_i \boldsymbol{x}_i\right) \leq L^{\delta} h(\boldsymbol{\xi}) \prod_{i=1}^{L} h(\boldsymbol{x}_i),$$

where $\boldsymbol{\xi} = (\xi_1, ..., \xi_L) \in K^L$, and δ is as in (28) above.

The second one is an adaptation of Lemma 4.7 of [17] to our choice of height functions, using (25).

Lemma 2.2. *Let V be a subspace of K^N, $N \geq 2$, and let subspaces $U_1, \dots, U_n \subseteq V$ and vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m \in V$ be such that*

$$V = \operatorname{span}_K\{U_1, \dots, U_n, \boldsymbol{x}_1, \dots, \boldsymbol{x}_m\}.$$

Then

$$\mathcal{H}(V) \leq N^{\delta m/2} \mathcal{H}(U_1) \dots \mathcal{H}(U_n) H(\boldsymbol{x}_1) \dots H(\boldsymbol{x}_m),$$

where δ is as in (28) above.

The next one is an adaptation of Lemma 2.3 of [9] to our choice of height functions, using (25).

Lemma 2.3. *Let X be a $J \times N$ matrix over K with row vectors $\boldsymbol{x}_1, ..., \boldsymbol{x}_J$, and let F be a symmetric bilinear form in N variables over K (we also write F for its*

$N \times N$ coefficient matrix). Then

$$\mathcal{H}(XF) \leq N^{3J\delta/2} H(F)^J \prod_{i=1}^{J} H(\boldsymbol{x}_i),$$

where δ is as in (28) above.

The next one is Lemma 2.2 of [9] over any global field.

Lemma 2.4. *Let U_1 and U_2 be subspaces of K^N. Then*

$$\mathcal{H}(U_1 \cap U_2) \leq \mathcal{H}(U_1)\mathcal{H}(U_2).$$

Remark 2.2. Lemmas 2.1 - 2.4 also hold verbatim with K replaced by $\overline{K}$. In fact, we will use them in both situations.

Finally, we introduce some basic language of quadratic forms (see, for instance, Chapter 1 of [18] for an introduction into the subject). We write

$$F(\boldsymbol{X}, \boldsymbol{Y}) = \sum_{i=1}^{N} \sum_{j=1}^{N} f_{ij} X_i Y_j$$

for a symmetric bilinear form in $2N$ variables with coefficients $f_{ij} = f_{ji}$ in K, and $F(\boldsymbol{X}) = F(\boldsymbol{X}, \boldsymbol{X})$ for the associated quadratic form in N variables; we also use F to denote the symmetric $N \times N$ coefficient matrix $(f_{ij})_{1 \leq i,j \leq N}$. We will write $H(F)$ for the height of F, which is just the height of its coefficient vector, as specified above for polynomials. Let $V \subseteq K^N$ be an L-dimensional subspace, $2 \leq L \leq N$, then F is also defined on V, and we write (V, F) for the corresponding quadratic space.

A point $\boldsymbol{x}$ in a subspace U of V is called singular if $F(\boldsymbol{x}, \boldsymbol{y}) = 0$ for all $\boldsymbol{y} \in U$, and it is called non-singular otherwise. For each subspace U of (V, F) we define its radical

$$U^{\perp} := \{\boldsymbol{x} \in U : F(\boldsymbol{x}, \boldsymbol{y}) = 0 \ \forall \ \boldsymbol{y} \in U\}$$

to be the space of all singular points in U. We define $\lambda(U) := \dim_K U^{\perp}$, and will write λ to denote $\lambda(V)$. A subspace U of (V, F) is called regular if $\lambda(U) = 0$.

A point $\boldsymbol{0} \neq \boldsymbol{x} \in V$ is called isotropic if $F(\boldsymbol{x}) = 0$ and anisotropic otherwise. A subspace U of V is called isotropic if it contains an isotropic point, and it is called anisotropic otherwise. A totally isotropic subspace W of (V, F) is a subspace such that for all $\boldsymbol{x}, \boldsymbol{y} \in W$, $F(\boldsymbol{x}, \boldsymbol{y}) = 0$. All maximal totally isotropic subspaces of (V, F) contain $V^{\perp}$ and have the same dimension. Given any maximal totally isotropic subspace W of V, we define the Witt index of (V, F) to be

$$\omega = \omega(V) := \dim_K(W) - \lambda.$$

If $K = \overline{K}$, then $\omega = [(L - \lambda)/2]$, where [ ] stands for the integer part function.

If two subspaces U_1 and U_2 of (V, F) are orthogonal, we write $U_1 \perp U_2$ for their orthogonal sum. If U is a regular subspace of (V, F), then $V = U \perp (\perp_V (U))$ and $U \cap (\perp_V (U)) = \{\boldsymbol{0}\}$, where

$$\perp_V (U) := \{\boldsymbol{x} \in V : F(\boldsymbol{x}, \boldsymbol{y}) = 0 \ \forall \ \boldsymbol{y} \in U\} \tag{29}$$

is the orthogonal complement of U in V. Two vectors $\boldsymbol{x}, \boldsymbol{y} \in V$ are called a hyperbolic pair if $F(\boldsymbol{x}) = F(\boldsymbol{y}) = 0$ and $F(\boldsymbol{x}, \boldsymbol{y}) \neq 0$; the subspace $\mathbb{H}(\boldsymbol{x}, \boldsymbol{y}) := \operatorname{span}_K\{\boldsymbol{x}, \boldsymbol{y}\}$ that they generate is regular and is called a hyperbolic plane. An orthogonal sum of hyperbolic planes is called a hyperbolic space. Every hyperbolic space is regular. It is well known that there exists an orthogonal Witt decomposition of the quadratic space (V, F) of the form

$$V = V^{\perp} \perp \mathbb{H}_1 \perp \cdots \perp \mathbb{H}_\omega \perp U, \tag{30}$$

where $\mathbb{H}_1, \dots, \mathbb{H}_\omega$ are hyperbolic planes and U is an anisotropic subspace, which is determined uniquely up to isometry. The rank of F on V is $r := L - \lambda$. In case $K = \overline{K}$, $\dim_K U = 1$ if r is odd and 0 if r is even. Therefore a regular even-dimensional quadratic space over $\overline{\mathbb{Q}}$ is always hyperbolic.

We are now ready to proceed.

3. Quadratic forms over function fields

In this section we establish an analogue of Vaaler's result [21] on small-height isotropic subspaces of a quadratic space over function fields of finite type. We then use it to derive an effective version of Witt decomposition over such fields, which is an analogue of the corresponding results of [9] and [10].

For the rest of this section, let p be an odd prime and q a power of p. Let $\mathbb{F}_q$ be a finite field of q elements, and let K be a finite separable algebraic extension of $\mathfrak{K} = \mathbb{F}_q(t)$, as defined in Section 2 above. Let $v \in M(K)$, and let $\mathfrak{X} \subseteq K_v^N$ be an L-dimensional subspace, $1 \leq L \leq N$. Recall that since K is a function field, all of its places are non-archimedean. Let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_L$ be a basis for $\mathfrak{X}$ and $X = (\boldsymbol{x}_1 \ \dots \ \boldsymbol{x}_L)$ the corresponding $N \times L$ basis matrix. For a subset I of $\{1, \dots, N\}$ of cardinality L, define ${}_IX$ to be the $L \times L$ submatrix of X consisting of the rows indexed by I. Let $J = J_v$ be such a subset so that $|\det({}_JX)|_v$ is maximal. We define a matrix $\mathcal{P}_v = \mathcal{P}_v(X)$ as in (4.3) of [21]:

$$\mathcal{P}_v = X({}_JX)^{-1} \ {}_J(1_N), \tag{31}$$

where 1_N is the $N \times N$ identity matrix. As indicated in [21], $\mathcal{P}_v$ depends only on $\mathfrak{X}$ and not on the choice of a basis matrix X. In addition, equations (4.4), (4.5), and (4.6) of [21] establish that the matrix $\mathcal{P}_v$ acts as a projection operator from K_v^N onto $\mathfrak{X}$ (by left multiplication) which fixes $\mathfrak{X}$ point-wise and has the property $\mathcal{P}_v^2 = \mathcal{P}_v$. With this notation, we can now state some further useful properties of $\mathcal{P}_v$, all established in [21]. Throughout this section we keep in mind that the heights H and $\mathcal{H}$ are the same over a function field, since there are no archimedean places, and therefore we will use H to denote all projective heights.

Lemma 3.1. *Let $Q_v = \frac{1}{2}(1_N + \mathcal{P}_v)$, then:*

(i) $H_v(\boldsymbol{x}) = \max\{H_v(\mathcal{P}_v\boldsymbol{x}), H_v((1_N - \mathcal{P}_v)\boldsymbol{x})\}$ *for any* $\boldsymbol{x} \in K_v^N$,
(ii) $H_v(\boldsymbol{x}) = H_v(Q_v\boldsymbol{x})$ *for any* $\boldsymbol{x} \in K_v^N$,
(iii) $H_v(Y) = H_v(Q_vY)$ *for any $N \times L$ matrix Y over K_v with* $1 \leq \operatorname{rank}(Y) = L \leq N$.

Proof. See Lemma 7 (ii), Lemma 8 (iii), and Lemma 9 (iii) of [21]. □

Lemma 3.2. [21, Lemma 4] *Let X be a matrix whose columns form a basis of $\mathfrak{X}$, as above, and let M be an integer such that $L < M \leq N$. If Y is an $N \times (M - L)$ matrix over K_v so that $C = (X\ Y)$ is an $N \times M$ matrix of rank M, then*

$$H_v(C) = H_v(X)H_v((1_N - P_v)Y).$$

It should be remarked that in [21] Vaaler works over number fields, not function fields, however the proofs of all the results from [21] that we reference here carry over word for word.

Next, we will need an adelic version of Minkowski's successive minima theorem over function fields of finite type, as established in [19]. For each $v \in M(K)$, let μ_v be the Haar measure on K_v such that $\mu_v(\mathfrak{O}_v) = 1$. Let $K_{\mathbb{A}}$ be the ring of adeles of K. We choose a Haar measure μ on $K_{\mathbb{A}}$ which is given by

$$\mu_K = q^{1-g(K)} \prod_v \mu_v,$$

where $g(K)$ is the genus of K. We denote the corresponding product measures on K_v^N and $K_{\mathbb{A}}^N$ by μ_v^N and μ_K^N respectively.

A measurable subset $\mathcal{S} \subseteq K_{\mathbb{A}}^N$ is called a *star body* if $0 \in \mathcal{S}$ and, for every $\boldsymbol{x} \in \mathcal{S}$, $a\boldsymbol{x}$ is an interior point of $\mathcal{S}$ whenever $a = (a_v)_{v \in M(K)} \in K_{\mathbb{A}}^\times$ satisfies $|a_v|_v \leq 1$ for all $v \in M(K)$. For an $A \in \mathrm{GL}_N(K_{\mathbb{A}})$, we define the successive minima of $\mathcal{S}$ with respect to A as in [19]:

$$\lambda_i(\mathcal{S}, A) = \inf_{a \in K_{\mathbb{A}}^\times} \left\{ |a|_{\mathbb{A}} : \dim_K \left(\operatorname{span}_K \left(a\mathcal{S} \cap AK^N \right) \right) \geq i \right\},$$

for each $i = 1, \dots, N$, where $|a|_{\mathbb{A}} := \prod_v |a_v|_v$ for each $a = (a_v) \in K_{\mathbb{A}}$.

A star body $\mathcal{S}$ is called a *coherent system of $\mathfrak{O}_v$-lattices* (see p. 97 of [23]) if $\mathcal{S} = \prod_v \mathcal{S}_v$ such that each $\mathcal{S}_v$ is an $\mathfrak{O}_v$-lattice on K_v^N and $\mathcal{S}_v = \mathfrak{O}_v^N$ for all but finitely many v. With this notation, we have the following adelic analogue of Minkowski's second convex body theorem.

Theorem 3.3. *Let $\mathcal{S}$ be a coherent system of $\mathfrak{O}_v$-lattices in $K_{\mathbb{A}}^N$, and let $A \in \mathrm{GL}_N(K_{\mathbb{A}})$. Then*

$$\mu_{K^N}(\mathcal{S}) \prod_{i=1}^N \lambda_i(\mathcal{S}, A) \leq q^N |\det(A)|_{\mathbb{A}}.$$

Proof. This is essentially Corollary 1 of [19], even though only the case $\mathcal{S} = \prod_v \mathfrak{O}_v^N$ is proved there. For each $v \in M(K)$, let $B_v \in \mathrm{GL}_N(K_v)$ be such that $B_v \mathcal{S}_v = \mathfrak{O}_v^N$. For all but finitely many v, $\mathcal{S}_v = \mathfrak{O}_v^N$, and hence $|\det(B_v)|_v = 1$. Thus $B = (B_v) \in \mathrm{GL}_N(K_{\mathbb{A}})$. It is clear that $\lambda_i(\mathcal{S}, A) = \lambda_i(\prod_v \mathfrak{O}_v^N, BA)$ for all i. On the other hand, $\mu_{K^N}(\prod_v \mathfrak{O}_v^N) = \mu_{K^N}(B\mathcal{S}) = |\det(B)|_{\mathbb{A}} \mu_{K^N}(\mathcal{S})$. The result then follows immediately from Corollary 1 of [19]. □

Next let F be a nonzero quadratic form in $K[X_1, \dots, X_N]$ and let $V \subseteq K^N$ be an L-dimensional vector space, $1 \leq L \leq N$. Suppose that the quadratic space (V, F) has a totally isotropic subspace of dimension $\ell \geq 1$. Among all the totally isotropic subspaces of dimension ℓ in V, let $\mathcal{A}$ be one with the smallest height (such a subspace exists by the Northcott finiteness property, since K is a function field of finite type). For each $v \in M(K)$, let $\mathcal{P}_v = \mathcal{P}_v(\mathcal{A})$ be the projection of K_v^N onto $\mathcal{A}_v$, the completion of $\mathcal{A}$ at v, as defined in (31) above. We will assume throughout that ℓ is strictly larger than the dimension of $V^\perp$, the radical of (V, F). Then $\perp_V(\mathcal{A})$, defined as in (29), is a proper subspace of V: otherwise $\mathcal{A} \subseteq V^\perp$, which is not possible by comparing their dimensions.

Proposition 3.4. *Let $\boldsymbol{b}$ be a vector in $V \setminus (\perp_V(\mathcal{A}))$ and let $\mathcal{B}$ be the $(\ell+1)$-dimensional subspace of V spanned by $\mathcal{A}$ and $\boldsymbol{b}$. Then there exists an ℓ-dimensional totally isotropic subspace $\mathcal{A}' \subseteq \mathcal{B}$ such that*

(i) $\dim(\mathcal{A} \cap \mathcal{A}') = \ell - 1$,
(ii) $H(\mathcal{A})^2 \leq H(\mathcal{A})H(\mathcal{A}') \leq H(F)H(\mathcal{B})^2$.

In addition, the following inequality is satisfied:

(iii) $1 \leq H(F) \prod_{v \in M(K)} H_v((1_N - \mathcal{P}_v)\boldsymbol{b})^{2/d}$.

Proof. Since $\boldsymbol{b} \notin \perp_V (\mathcal{A})$, the quadratic space $(\mathcal{B}, F)$ is isometric to the orthogonal sum of a hyperbolic plane and a radical of dimension $\ell - 1$. Therefore, $\mathcal{B}$ has exactly two ℓ-dimensional totally isotropic subspaces. One is $\mathcal{A}$, and we call the other $\mathcal{A}'$. Note that $\mathcal{A} \cap \mathcal{A}'$ is the radical of $\mathcal{B}$, which implies (i). For each $v \in M(K)$, let $\mathcal{P}'_v$ be the projection of K_v^N onto $\mathcal{A}'_v$, the completion of $\mathcal{A}'$ at v, as defined in (31) above.

Now we select a vector $\boldsymbol{y} \in \mathcal{B} \setminus (\mathcal{A} \cup \mathcal{A}')$. Then $\boldsymbol{y}$ is anisotropic and $\boldsymbol{y} = \boldsymbol{a} + \beta\boldsymbol{b}$ for some $\boldsymbol{a} \in \mathcal{A}$ and $\beta \in K^\times$. Following the argument in [22, Page 679], we see that equation (4.7) of [22] implies

$$0 \neq F(\boldsymbol{y}) = 2\beta F(Q_v(1_N - \mathcal{P}'_v)\boldsymbol{y}, (1_N - \mathcal{P}_v)\boldsymbol{b}),$$

and hence by Lemma 3.1

$$\begin{aligned} 1 &= \prod_v |(2\beta)^{-1} F(\boldsymbol{y})|_v^{1/d} \\ &\leq H(F) \left\{ \prod_v H_v(Q_v(1_N - \mathcal{P}'_v)\boldsymbol{y}) \right\}^{1/d} \left\{ \prod_v H_v((1_N - \mathcal{P}_v)\boldsymbol{b}) \right\}^{1/d} \\ &\leq H(F) \left\{ \prod_v H_v((1_N - \mathcal{P}'_v)\boldsymbol{y}) \right\}^{1/d} \left\{ \prod_v H_v((1_N - \mathcal{P}_v)\boldsymbol{b}) \right\}^{1/d}. \end{aligned}$$

Multiplying both sides of the inequality by $H(\mathcal{A})H(\mathcal{A}')$ and applying Lemma 3.2 we obtain (ii) and (iii). □

Remark 3.1. Notice that the proof of part (iii) of Proposition 3.4 holds when $\mathcal{A}$ is any totally isotropic subspace of (V, F). We will use (iii) in this more general context in the proof of Lemma 3.9 below.

Proposition 3.5. *Let $U \subset V$ be an m-dimensional subspace, $1 \leq m < L$. For each $v \in M(K)$, let $\mathcal{P}_v(U)$ be the projection of K_v^N onto U_v, the completion of U at v, as defined in* (31) *above. There exist $L - m$ linearly independent vectors $\boldsymbol{b}_1, \dots, \boldsymbol{b}_{L-m}$ in a (vector space) complement of U in V such that*

$$\prod_{i=1}^{L-m} \prod_v H_v((1_N - \mathcal{P}_v(U))\boldsymbol{b}_i)^{1/d} \leq q^{(L-m)g(K)/d} \left(\frac{H(V)}{H(U)} \right).$$

Proof. Let Y be an $N \times (L - m)$ matrix whose columns form a basis of a (vector space) complement of U in V. For each $v \in M(K)$, let

$$\mathcal{S}_v = \{\boldsymbol{u} \in K_v^{L-m} : (1_N - \mathcal{P}_v(U))Y\boldsymbol{u} \in \mathfrak{O}_v^N\}.$$

The $N \times (L - m)$ matrix $T_v := (1_N - \mathcal{P}_v(U))Y$ has rank $L - \ell$. For, if $(1_N - \mathcal{P}_v(U))Y\boldsymbol{u} = \boldsymbol{0}$, then $Y\boldsymbol{u} \in U$, which implies that $\boldsymbol{u} = \boldsymbol{0}$. Moreover,

$$H_v(V) = H_v(U)H_v(T_v)$$

by Lemma 3.2.

Fix $v \in M(K)$. By rearranging the coordinates if necessary, we may assume that $|\det({}_IT_v)|_v = H_v(T_v)$ where $I = \{1, \ldots, L-m\}$. Then $T_v({}_IT_v^{-1})$ has the identity matrix 1_{L-m} on top and the entries below are all in $\mathfrak{O}_v$. This shows that ${}_IT_v^{-1}\mathfrak{O}_v^{L-m} = \mathcal{S}_v$, and so $\mathcal{S}_v = \mathfrak{O}_v^{L-m}$ for almost all v. Then $\mathcal{S} := \prod_v \mathcal{S}_v$ is a coherent system of $\mathfrak{O}_v$-lattices in $K_{\mathbb{A}}^{L-m}$. Moreover,

$$\mu_v^{L-m}(\mathcal{S}_v) = |\det({}_IT_v)|_v^{-1} = H_v((1_N - \mathcal{P}_v(U))Y)^{-1}.$$

As a result,

$$\mu_K^{L-m}(\mathcal{S}) = q^{(L-m)(1-g(K))} \prod_v H_v((1_N - \mathcal{P}_v(U))Y)^{-1} = q^{(L-m)(1-g(K))} \left(\frac{H(U)}{H(V)}\right)^d.$$

By Theorem 3.3, the successive minima $\lambda_1, \ldots, \lambda_{L-m}$ of $\mathcal{S}$ with respect to the identity element $(1_{L-m})_{v \in M(K)} \in \mathrm{GL}_{L-m}(K_{\mathbb{A}})$ satisfy

$$\lambda_1 \cdots \lambda_{L-m} \leq q^{L-m} \mu_K^{L-m}(\mathcal{S})^{-1} = q^{(L-m)g(K)} \left(\frac{H(V)}{H(U)}\right)^d.$$

Let $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{L-m}\}$ be a set of linear independent vectors associated with the successive minima. In particular, for $i = 1, \ldots, L-m$, we have

$$H_v((1_N - \mathcal{P}_v(U))Y\boldsymbol{u}_i) \leq \lambda_i.$$

The proposition now follows by setting $\boldsymbol{b}_i = Y\boldsymbol{u}_i$ for $i = 1, \ldots, L-m$. □

Corollary 3.6. *There exists a nonzero vector $\boldsymbol{b} \in V$ such that*

(i) *the subspace $\mathcal{B} := \mathrm{span}_K\{\mathcal{A}, \boldsymbol{b}\} \subseteq V$ has dimension $\ell + 1$,*

(ii) *the vectors $(1_N - \mathcal{P}_v)\boldsymbol{b}$, $v \in M(K)$, satisfy*

$$\prod_v H_v((1_N - \mathcal{P}_v)\boldsymbol{b})^{1/d} \leq q^{(L-\ell)g(K)/d} \left(\frac{H(V)}{H(\mathcal{A})}\right)^{1/(L-\ell)},$$

(iii) *the subspace $\mathcal{B}$ satisfies*

$$H(\mathcal{B}) \leq q^{(L-\ell)g(K)/d} H(\mathcal{A})^{1-1/(L-\ell)} H(V)^{1/(L-\ell)}.$$

Proof. Let $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{L-\ell}$ be the vectors obtained in Proposition 3.5 with $U = \mathcal{A}$ and $m = \ell$, and arrange them so that

$$\prod_v H_v((1_N - \mathcal{P}_v)\boldsymbol{b}_1) \leq \prod_v H_v((1_N - \mathcal{P}_v)\boldsymbol{b}_2) \leq \ldots \leq \prod_v H_v((1_N - \mathcal{P}_v)\boldsymbol{b}_{L-\ell}).$$

Then set $\boldsymbol{b} = \boldsymbol{b}_1$. Statements (i) and (ii) are then clear, while statement (iii) follows by a direct application of Lemma 3.2. □

Proof of Theorem 1.1. Combine Proposition 3.4 (ii) with Corollary 3.6 (iii). □

Using the function field version of Siegel's lemma [20, Corollary 2], we deduce the following results on zeros of F of small height.

Corollary 3.7. *Let $\mathcal{A}$ be the subspace of V obtained in Theorem 1.1. There exists a basis $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_\ell$ of $\mathcal{A}$ such that*

$$\prod_{i=1}^{\ell} H(\boldsymbol{x}_i) \leq q^{(L^2-\ell+\ell^2)g(K)/d} H(F)^{(L-\ell)/2} H(V). \tag{32}$$

In particular, if l is the dimension of a maximal totally isotropic subspace of V, then V contains an isotropic vector $\boldsymbol{a}$ satisfying

$$H(\boldsymbol{a}) \leq q^{(L^2-\ell+\ell^2)g(K)/\ell d} H(F)^{(L-\ell)/2\ell} H(V)^{1/\ell}. \tag{33}$$

Proof. Inequality (32) follows by combining Corollary 2 of [20] with our Theorem 1.1. Now (33) follows from (32) by taking $\mathcal{A}$ to be a maximal totally isotropic subspace of small height (i.e., $\ell = \omega$) and letting $\boldsymbol{a}$ be the vector of smallest height among $\boldsymbol{x}_1, \dots, \boldsymbol{x}_\omega$. □

Under our assumptions on the quadratic space (V, F), every maximal totally isotropic subspace of V must properly contain the radical of (V, F). In other words, (V, F) always contains a nonsingular vector.

Corollary 3.8. *There exists a nonsingular isotropic vector $\boldsymbol{a} \in V$ such that*

$$H(\boldsymbol{a}) \leq h(\boldsymbol{a}) \leq q^{(2L^2-3L+2)g(K)/d} H(F)^{(L-1)/2} H(V).$$

Proof. In Corollary 3.7, one of the $\boldsymbol{x}_i$'s must be nonsingular. Then one of the coordinates of $\boldsymbol{x}_i$ must be nonzero, say $x_{ij} \neq 0$ for some $1 \leq j \leq N$. Define $\boldsymbol{a} = \frac{1}{x_{ij}} \boldsymbol{x}_i$, then $\boldsymbol{a}$ is again a nonsingular zero of F in V, one of which coordinates is equal to 1. Hence

$$h(\boldsymbol{a}) = H(\boldsymbol{a}) = H(\boldsymbol{x}_i),$$

by the product formula. Then the corollary follows immediately from Corollary 3.7, as $1 \leq \ell \leq L - 1$. □

With notation as above, we are now ready to produce an effective version of Witt decomposition theorem for a quadratic space over a function field of finite type. We start by obtaining a bound on the height of the radical of a quadratic space.

Lemma 3.9. *Suppose that the quadratic space (V, F) has rank $1 \leq r < L$. Then*

$$H(V^{\perp}) \leq q^{rg(K)/d} H(F)^{r/2} H(V). \tag{34}$$

Proof. Our argument is identical to the proof of Theorem 2 of [22], using our Proposition 3.5 instead of Theorem 10 of [21]. Since the rank of (V, F) is $r < L$, there must exist an r-dimensional subspace W of V such that F is nonsingular on W, i.e., W is contained in the vector space complement of $V^{\perp}$ in V. By Proposition 3.5 there exists a basis $\boldsymbol{b}_1, \dots, \boldsymbol{b}_r$ for W such that

$$\prod_{i=1}^{r} \prod_{v} H_v((1_N - \mathcal{P}_v(V^{\perp}))\boldsymbol{b}_i)^{1/d} \leq q^{rg(K)/d} \left(\frac{H(V)}{H(V^{\perp})} \right). \tag{35}$$

Now the statement of the lemma follows readily by combining Proposition 3.4 (iii) (see also Remark 3.1) with inequality (35). □

Now that we have a bound on the height of the radical of a quadratic space, it is sufficient to obtain an effective Witt decomposition for a regular space over a function field of finite type. Our argument follows the proof of Theorem 1.3 of [9], replacing Vaaler's theorem on small-height maximal totally isotropic subspaces over a number field with our Theorem 1.1 and Bombieri-Vaaler Siegel's lemma [1] with Thunder's function fields version of Siegel's lemma [20].

Theorem 3.10. *Suppose that the (V,F) is an L-dimensional regular quadratic space, i.e., $V^{\perp} = \{\boldsymbol{0}\}$, of Witt index $\omega \geq 1$ in $N \geq L \geq 1$ variables over the function field K of finite type, as above. There exists an orthogonal decomposition of (V,F) of the form*

$$V = \mathbb{H}_1 \perp \ \dots \ \perp \mathbb{H}_{\omega} \perp U, \tag{36}$$

where $\mathbb{H}_1, \dots, \mathbb{H}_{\omega}$ are hyperbolic planes, U is anisotropic, and

$$\max\{H(\mathbb{H}_i), H(U)\} \leq \mathcal{G}_K(N,L,\omega) \left\{ H(F)^{\frac{L+2\omega}{4}} H(V) \right\}^{\frac{(\omega+1)(\omega+2)}{2}}, \tag{37}$$

for each $1 \leq i \leq \omega$, where

$$\mathcal{G}_K(N,L,\omega) = \left\{ C_K(L) q^{\frac{(L^2-\omega+\omega^2)g(K)}{d}} \right\}^{\frac{\omega(\omega+1)}{4}} \tag{38}$$

and $C_K(L)$ is as in (19).

Proof. Let $\mathcal{A}$ be a maximal totally isotropic subspace of (V,F) satisfying (1) of Theorem 1.1 with $\ell = \omega$ and let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{\omega}$ be the basis for $\mathcal{A}$ guaranteed by (32) of Corollary 3.7 above. Notice that $F(\boldsymbol{x}_i, \boldsymbol{x}_j) = 0$ for all $1 \leq i, j \leq \omega$, since $\mathcal{A}$ is a totally isotropic subspace. Let $\boldsymbol{y}_1, \dots, \boldsymbol{y}_L$ be the basis for V guaranteed by the function field version of Siegel's lemma [20, Corollary 2] (see also [11, Theorem 1.1] for a convenient formulation), ordered so that

$$H(\boldsymbol{y}_1) \leq H(\boldsymbol{y}_2) \leq \dots \leq H(\boldsymbol{y}_L),$$

then

$$\prod_{i=1}^{L} H(\boldsymbol{y}_i) \leq C_K(L) H(V), \tag{39}$$

where $C_K(L)$ is as in (19). For each $1 \leq i \leq \omega$ let j_i be the smallest index such that $F(\boldsymbol{x}_i, \boldsymbol{y}_{j_i}) \neq 0$. Such j_i exists for each i since otherwise $\boldsymbol{x}_i$ would be a singular point, contradicting regularity of (V,F). By reordering $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{\omega}$ if necessary, we can assume without loss of generality that

$$1 \leq j_{\omega} \leq j_{\omega-1} \leq \dots \leq j_1 \leq L.$$

Moreover, for each $1 \leq i \leq \omega$, $j_i \leq L - i + 1$, since

$$\operatorname{span}_K\{\boldsymbol{y}_1, \dots, \boldsymbol{y}_{L-i+1}\} \nsubseteq \operatorname{span}_K\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_i\}^{\perp},$$

and so $H(\boldsymbol{y}_{j_i}) \leq H(\boldsymbol{y}_{L-i+1})$ by our ordering of $\boldsymbol{y}_1, \dots, \boldsymbol{y}_L$. Therefore, combining (32) and (39), we obtain

$$\begin{aligned} \prod_{i=1}^{\omega} H(\boldsymbol{x}_i) H(\boldsymbol{y}_{j_i}) &\leq \prod_{i=1}^{\omega} H(\boldsymbol{x}_i) H(\boldsymbol{y}_{L-i+1}) \\ &= \left(\prod_{i=1}^{\omega} H(\boldsymbol{x}_i) \right) \left(\prod_{i=1}^{\omega} H(\boldsymbol{y}_{L-i+1}) \right) \\ &\leq C_K(L) q^{(L^2-\omega+\omega^2)g(K)/d} H(F)^{(L-\omega)/2} H(V)^2. \end{aligned} \tag{40}$$

In particular, for some $1 \leq i \leq \omega$, we must have

$$H(\boldsymbol{x}_i) H(\boldsymbol{y}_{j_i}) \leq \left\{ C_K(L) q^{\frac{(L^2-\omega+\omega^2)g(K)}{d}} H(F)^{\frac{L-\omega}{2}} H(V)^2 \right\}^{\frac{1}{\omega}}. \tag{41}$$

Define $\mathbb{H}_1 = \operatorname{span}_K\{\boldsymbol{x}_i, \boldsymbol{y}_{j_i}\}$ for this choice of i. Since $F(\boldsymbol{x}_i) = 0$ and $F(\boldsymbol{x}_i, \boldsymbol{y}_{j_i}) \neq 0$, $\mathbb{H}_1$ is a regular subspace of Z with Witt index equal to one, hence it is a hyperbolic plane. Notice that Lemma 2.2 implies that

$$H(\mathbb{H}_1) \leq H(\boldsymbol{x}_i)H(\boldsymbol{y}_{j_i}). \tag{42}$$

Let V_1 be the orthogonal complement of $\mathbb{H}_1$ in V. Notice that

$$V_1 = \perp_V (\mathbb{H}_1) = \{\boldsymbol{z} \in K^N : F(\boldsymbol{z}, \boldsymbol{x}) = 0 \ \forall \ \boldsymbol{x} \in \mathbb{H}_1\} \cap V,$$

so $\dim_K(V_1) = L - 2$, and $V = \mathbb{H}_1 \perp V_1$. Notice that by combining Lemma 2.4, Lemma 2.3, and (42), and (41), we have

$$\begin{aligned} H(V_1) &\leq H(\mathbb{H}_1)H(V)H(F)^2 \leq H(\boldsymbol{x}_i)H(\boldsymbol{y}_{j_i})H(V)H(F)^2 \\ &\leq \left\{ C_K(L) q^{\frac{(L^2-\omega+\omega^2)g(K)}{d}} \right\}^{\frac{1}{\omega}} H(F)^{\frac{L+3\omega}{2\omega}} H(V)^{\frac{\omega+2}{\omega}}. \end{aligned} \tag{43}$$

We continue by induction on ω. If $\omega = 1$, we are done. If $\omega \geq 2$, assume that the theorem holds for a bilinear space of Witt index smaller than ω, in particular it holds for (V_1, F), a bilinear space of dimension $L - 2$ and Witt index $\omega - 1$. Then there exists a decomposition

$$V_1 = \mathbb{H}_2 \perp \ \dots \ \perp \mathbb{H}_\omega \perp U, \tag{44}$$

where U, the anisotropic component of V_1, is the same as that of V. Combining the induction hypothesis with (43), for each $2 \leq i \leq \omega$ we obtain

$$\begin{aligned} \max\{H(\mathbb{H}_i), H(U)\} &\leq \mathcal{G}_K(N, L-2, \omega-1) \left\{ H(F)^{\frac{L+2\omega-4}{4}} H(V_1) \right\}^{\frac{\omega(\omega+1)}{2}} \\ &\leq \mathcal{G}_K(N, L-2, \omega-1) \left\{ C_K(L) q^{\frac{(L^2-\omega+\omega^2)g(K)}{d}} \right\}^{\frac{\omega+1}{2}} \times \\ &\times \left\{ H(F)^{\frac{L+2\omega-4}{4} + \frac{L+3\omega}{2\omega}} H(V)^{\frac{\omega+2}{\omega}} \right\}^{\frac{\omega(\omega+1)}{2}} \\ &\leq \mathcal{G}_K(N, L, \omega) \left\{ H(F)^{\frac{L+2\omega}{4}} H(V) \right\}^{\frac{(\omega+1)(\omega+2)}{2}}. \end{aligned} \tag{45}$$

This finishes the proof. □

Corollary 3.11. *Suppose that (V, F) is an L-dimensional quadratic space of Witt index $\omega \geq 1$ in $N \geq L \geq 1$ variables over the function field K of finite type, as above. There exists an orthogonal decomposition of (V, F) of the form*

$$V = V^{\perp} \perp \mathbb{H}_1 \perp \ \dots \ \perp \mathbb{H}_\omega \perp U, \tag{46}$$

where $V^{\perp}$ is the radical, $\mathbb{H}_1, \dots, \mathbb{H}_\omega$ are hyperbolic planes, and U is anisotropic component so that $H(V^{\perp})$ is bounded as in (34) *of Lemma 3.9 above and*

$$\max\{H(\mathbb{H}_i), H(U)\} \leq \mathcal{G}_K(N, L, \omega) \left\{ C_K(L) H(F)^{\frac{L+2\omega}{4}} H(V) \right\}^{\frac{(\omega+1)(\omega+2)}{2}}, \tag{47}$$

for each $1 \leq i \leq \omega$, where $\mathcal{G}_K(N, L, \omega)$ is as in (38) *and $C_K(L)$ is as in* (19) *above.*

Proof. If (V, F) is regular, then $V^{\perp} = \{\boldsymbol{0}\}$, and we are done by Theorem 3.10. Let r be rank of F on V, and assume that $1 \leq r < L$. Then the quadratic space (V, F) can be represented as

$$V = V^{\perp} \perp R, \tag{48}$$

where R is a regular subspace of V, with $H(V^{\perp})$ bounded as in (34) of Lemma 3.9 above and

$$H(R) \leq C_K(L) H(V). \tag{49}$$

Indeed, let $\boldsymbol{y}_1, \dots, \boldsymbol{y}_L$ be the basis for V guranteed by (39) above, and notice that $\dim_K(V^{\perp}) = L - r$. We can now pick r vectors $\boldsymbol{y}_{i_1}, \dots, \boldsymbol{y}_{i_r}$ from our basis for V such that

$$\operatorname{span}_K\{V^{\perp}, \boldsymbol{y}_{i_1}, ..., \boldsymbol{y}_{i_r}\} = V.$$

Let $R = \operatorname{span}_K\{\boldsymbol{y}_{i_1}, ..., \boldsymbol{y}_{i_r}\}$, which is regular and whose isometry class is uniquely determined by V. Then $V = V^{\perp} \perp R$ and Lemma 2.2 combined with (39) imply that

$$H(R) = H(\boldsymbol{y}_{i_1} \wedge \dots \wedge \boldsymbol{y}_{i_r}) \leq \prod_{j=1}^{r} H(\boldsymbol{y}_{i_j}) \leq C_K(L) H(V).$$

Now we can apply Theorem 3.10 to the regular quadratic space (R, F) and use (49). The result follows. □

4. Families of generating isotropic subspaces

In this section we use the effective Witt decomposition results of [9], [10], and Corollary 3.11 above to prove Theorem 1.2. The following is a combination of Theorem 1.3 of [9] with Lemma 3.5 and Theorem 5.1 of [10].

Theorem 4.1. *Let F be a symmetric bilinear form on K^N, where K is either a number field or $\overline{\mathbb{Q}}$. Let $V \subseteq K^N$ be a subspace of dimension L, $2 \leq L \leq N$, and Witt index $\omega \geq 1$. Let F have rank r on V, $1 \leq r \leq L$. There exists an orthogonal decomposition of the quadratic space (V, F) of the form (30) with all components of bounded height. Specifically,*

$$\mathcal{H}(V^{\perp}) \leq \begin{cases} B_K(r)^r H(F)^{r/2} \mathcal{H}(V) & \text{if } K \text{ is a number field} \\ 3^{\frac{L(L-1)}{2}} H(F)^r \mathcal{H}(V)^2 & \text{if } K = \overline{\mathbb{Q}}, \end{cases} \tag{50}$$

where the constant $B_K(r)$ is defined in (22) above. Moreover, if K is a number field, then

$$\max\{\mathcal{H}(\mathbb{H}_i), \mathcal{H}(U)\} \leq \mathcal{G}_K(N, L, \omega) \left\{ H(F)^{\frac{L+2\omega}{4}} \mathcal{H}(V) \right\}^{\frac{(\omega+1)(\omega+2)}{2}}, \tag{51}$$

for each $1 \leq i \leq \omega$, where

$$\mathcal{G}_K(N, L, \omega) = \left\{ \left(2^{2\omega+1} B_K(L)^2\right)^L \left(N |\mathcal{D}_K|^{1/d}\right)^{\omega+5L} \right\}^{\frac{\omega(\omega+3)}{8}}. \tag{52}$$

If $K = \overline{\mathbb{Q}}$, then

$$\mathcal{H}(\mathbb{H}_i) \leq 3^{12k^4(k+1)\left(\frac{3}{2}\right)^k} \left\{ \sqrt{k}\ H(F)^{k^2+1} \left(\eta(L, r) \mathcal{H}(V)\right)^{\frac{6k+5}{4k+2}} \right\}^{\frac{(k+1)(k+2)}{2}\left(\frac{3}{2}\right)^k}, \tag{53}$$

and $U = \{\boldsymbol{0}\}$ if $r = 2k$, or $U = \overline{\mathbb{Q}}\boldsymbol{y}$ with

$$\mathcal{H}(U) = \mathcal{H}(\boldsymbol{y}) \leq 2\sqrt{2k+1}\ 3^{\frac{(2k+3)k}{2}} \left(\eta(L, r) \mathcal{H}(V)\right)^{\frac{2k+3}{4k+2}}, \tag{54}$$

if $r = 2k+1$. *The constant* $\eta(L,r)$ *in equations* (53) *and* (54) *is defined by*

$$\eta(L,r) = \begin{cases} 3^{\frac{L(L-1)}{2}} & \text{if } r < L \\ 1 & \text{if } r = L. \end{cases}$$

We will also need a lemma on the existence of a small-height hyperbolic pair in a given hyperbolic plane.

Lemma 4.2. *Let* K *be a number field (abbreviated n.f.), function field of finite type* q *as in Section 3 above (abbreviated f.f.f), or* $\overline{\mathbb{Q}}$, *and let* F *be a symmetric bilinear form in* $2N$ *variables over* K. *Let* $\mathbb{H} \subseteq K^N$ *be a hyperbolic plane with respect to* F. *Then there exists a basis* $\boldsymbol{x}, \boldsymbol{y}$ *for* $\mathbb{H}$ *such that*

$$F(\boldsymbol{x}) = F(\boldsymbol{y}) = 0, \ F(\boldsymbol{x}, \boldsymbol{y}) \neq 0,$$

and

$$H(\boldsymbol{x}) \leq h(\boldsymbol{x}) \leq \begin{cases} 2\sqrt{2}\ B_K(1)^2 H(F)^{\frac{1}{2}} \mathcal{H}(\mathbb{H}) & \text{if } K \text{ n.f.} \\ q^{4g(K)/d} H(F)^{1/2} \mathcal{H}(\mathbb{H}), & \text{if } K \text{ f.f.f.} \\ 72\ H(F)^{\frac{1}{2}} \mathcal{H}(\mathbb{H})^2 & \text{if } K = \overline{\mathbb{Q}}, \end{cases} \tag{55}$$

as well as

$$H(\boldsymbol{y}) \leq h(\boldsymbol{y}) \leq \begin{cases} 24\sqrt{2} N^2\ \left(B_K(1)\mathcal{A}_K\right)^2 H(F)^{\frac{3}{2}} \mathcal{H}(\mathbb{H})^3 & \text{if } K \text{ n.f.} \\ 4q^{4g(K)/d} \mathcal{A}_K^2 H(F)^{\frac{3}{2}} \mathcal{H}(\mathbb{H})^3 & \text{if } K \text{ f.f.f.} \\ 864 N^2\ \mathcal{A}_K^2 H(F)^{\frac{3}{2}} \mathcal{H}(\mathbb{H})^4 & \text{if } K = \overline{\mathbb{Q}}, \end{cases} \tag{56}$$

where the constant $\mathcal{A}_K = \mathcal{E}_K(2) A_K(2) C_K(2)$.

Proof. The hyperbolic plane $(\mathbb{H}, F)$ is a regular 2-dimensional isotropic subspace of K^N, therefore Corollary 2 of [21] (when K is a number field), Corollary 3.7 above (when K is a function field), and Lemma 4.1 of [10] (when $K = \overline{\mathbb{Q}}$) imply the existence of $\boldsymbol{0} \neq \boldsymbol{x} \in \mathbb{H}$ such that $F(\boldsymbol{x}) = 0$ and height of $\boldsymbol{x}$ is bounded as in (55). Now Theorem 1.4 of [11] guarantees the existence of a point $\boldsymbol{z} \in \mathbb{H}$ such that $F(\boldsymbol{z}) \neq 0$ and

$$H(\boldsymbol{z}) \leq h(\boldsymbol{z}) \leq 2\mathcal{E}_K(2)^{1-\delta} A_K(2) C_K(2) \mathcal{H}(\mathbb{H}). \tag{57}$$

Since $F(\boldsymbol{x}) = 0$ and $F(\boldsymbol{z}) \neq 0$, it must be true that $\boldsymbol{x}$ and $\boldsymbol{z}$ are linearly independent, and hence span $\mathbb{H}$. Therefore we must have $F(\boldsymbol{x}, \boldsymbol{z}) \neq 0$, since $(\mathbb{H}, F)$ is regular. Then define

$$\boldsymbol{y} = F(\boldsymbol{z})\boldsymbol{x} - 2F(\boldsymbol{x}, \boldsymbol{z})\boldsymbol{z}.$$

Clearly, $\mathbb{H} = \operatorname{span}_K\{\boldsymbol{x}, \boldsymbol{y}\}$, and it is easy to check that $F(\boldsymbol{y}) = 0$. Once again, regularity of $(\mathbb{H}, F)$ implies that $F(\boldsymbol{x}, \boldsymbol{y}) \neq 0$, and so $\boldsymbol{x}, \boldsymbol{y}$ is a hyperbolic pair basis for $\mathbb{H}$. Finally, we need to produce an estimate on the height of $\boldsymbol{y}$. In case K is a number field or $\overline{\mathbb{Q}}$, Lemma 2.3 of [8] implies that

$$H(\boldsymbol{y}) \leq h(\boldsymbol{y}) \leq 3N^2 H(F) h(\boldsymbol{x}) h(\boldsymbol{z})^2. \tag{58}$$

If K is a function field, the argument in the proof of Lemma 2.3 of [8] implies that

$$H(\boldsymbol{y}) \leq h(\boldsymbol{y}) \leq H(F) h(\boldsymbol{x}) h(\boldsymbol{z})^2, \tag{59}$$

since K has no archimedean absolute values. Combining estimates of (58), (59) with (55) and (57) produces (56). $\square$

We are now ready for the main argument of this section.

Proof of Theorem 1.2. With notation as in Theorem 4.1 and Corollary 3.11, let $\boldsymbol{x}_i, \boldsymbol{y}_i$ be a small-height hyperbolic pair for the corresponding hyperbolic plane $\mathbb{H}_i$ in the effective Witt decomposition of Theorem 4.1 and Corollary 3.11 for V, $1 \leq i \leq \omega$, guaranteed by Lemma 4.2 above. We will always assume that $h(\boldsymbol{x}_i) \leq h(\boldsymbol{y}_i)$, $H(\boldsymbol{x}_i) \leq H(\boldsymbol{y}_i)$ for each $1 \leq i \leq \omega$. Let U be the small-height anisotropic component of V as in Theorem 4.1 and Corollary 3.11, then

$$J := \dim_K U = (L - \lambda) - 2\omega = r - 2\omega, \tag{60}$$

where $\lambda = \dim_K V^{\perp}$, $\omega \geq 1$ is the Witt index of the quadratic space (V, F), and $r = L - \lambda$ is the rank of F on V. In particular, $J = 0$ if $K = \overline{\mathbb{Q}}$ and $2 \mid r$. If $J > 0$, let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_J$ be the small-height basis for U, guaranteed by Siegel's lemma (see [1] and [17], conveniently formulated in Theorem 1.1 of [11], as well as Theorem 1.2 of [11]):

$$\prod_{k=1}^{J} H(\boldsymbol{u}_k) \leq \prod_{k=1}^{J} h(\boldsymbol{u}_k) \leq C_K(J) \mathcal{E}_K(J)^{1-\delta} \mathcal{H}(U), \tag{61}$$

where δ is as in (28). Define the set of pairs of indices

$$\mathcal{I}_{\omega J} = \begin{cases} \{(i,j) : 1 \leq i \leq \omega,\ 1 \leq j \leq J\} & \text{if } J > 0 \\ \{(i,j) : 1 \leq i \neq j \leq \omega\} & \text{if } J = 0, \end{cases}$$

and for each pair $(i,j) \in \mathcal{I}_{\omega J}$ define

$$\alpha_{ij} = \begin{cases} -\frac{F(\boldsymbol{u}_j)}{2F(\boldsymbol{x}_i, \boldsymbol{y}_i)} & \text{if } J > 0 \\ -\frac{F(\boldsymbol{x}_j, \boldsymbol{y}_j)}{F(\boldsymbol{x}_i, \boldsymbol{y}_i)} & \text{if } J = 0, \end{cases} \tag{62}$$

so that $\alpha_{ij} \neq 0$. For each integer $n \geq 1$, let

$$\xi_n = \begin{cases} n & \text{if } K \text{ is a number field} \\ t^n & \text{if } K \text{ is a function field} \\ e^{\frac{2\pi i}{n}} & \text{if } K = \overline{\mathbb{Q}}, \text{ where } i = \sqrt{-1}. \end{cases} \tag{63}$$

Now, for each pair $(i,j) \in \mathcal{I}_{\omega J}$ and each $n \geq 1$, define subspace W_{ij}^n of V by

$$W_{ij}^n = \operatorname{span}_K \left\{ V^{\perp}, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{i-1}, \boldsymbol{x}_{i+1}, \dots, \boldsymbol{x}_{\omega}, \boldsymbol{x}_i + \xi_n^2 \alpha_{ij} \boldsymbol{y}_i + \xi_n \boldsymbol{u}_j \right\} \tag{64}$$

when $J > 0$, and

$$\begin{aligned} W_{ij}^n \quad = \quad & \operatorname{span}_K \Big\{ V^{\perp}, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{i-1}, \boldsymbol{x}_{i+1}, \dots, \boldsymbol{x}_{j-1}, \\ & \boldsymbol{x}_{j+1}, \dots, \boldsymbol{x}_{\omega}, \boldsymbol{x}_i + \xi_n \boldsymbol{y}_j, \boldsymbol{x}_j + \xi_n \alpha_{ij} \boldsymbol{y}_i \Big\} \end{aligned} \tag{65}$$

when $J = 0$. Then $\dim_K W_{ij}^n = \omega + \lambda$, and we will now show that W_{ij}^n is totally isotropic, i.e., given $\boldsymbol{z} \in W_{ij}^n$, we will prove that $F(\boldsymbol{z}) = 0$. First assume that $J > 0$, then

$$\boldsymbol{z} = \boldsymbol{z}' + \sum_{k=1, k \neq i}^{\omega} a_k \boldsymbol{x}_k + a_i \left(\boldsymbol{x}_i + \xi_n^2 \alpha_{ij} \boldsymbol{y}_i + \xi_n \boldsymbol{u}_j \right),$$

where $\boldsymbol{z}' \in V^{\perp}$ and $a_k \in K$ for all $1 \leq k \leq \omega$. Recall that $\boldsymbol{z}', \boldsymbol{x}_1, \dots, \boldsymbol{x}_{\omega}, \boldsymbol{u}_j$ are all orthogonal to each other, and $\boldsymbol{y}_i$ is orthogonal to $\boldsymbol{z}'$ and all $\boldsymbol{x}_k$ with $k \neq i$, as well as $F(\boldsymbol{z}') = F(\boldsymbol{x}_k) = F(\boldsymbol{y}_k) = 0$ for all k. Then we have

$$\begin{aligned} F(\boldsymbol{z}) &= a_i^2 F \left(\boldsymbol{x}_i + \xi_n^2 \alpha_{ij} \boldsymbol{y}_i + \xi_n \boldsymbol{u}_j \right) \\ &= a_i^2 \left(2\xi_n^2 \alpha_{ij} F(\boldsymbol{x}_i, \boldsymbol{y}_i) + \xi_n^2 F(\boldsymbol{u}_i) \right) = 0. \end{aligned}$$

Next suppose that $J=0$, then

$$\boldsymbol{z}=\boldsymbol{z}'+\sum_{k=1,k\neq i,j}^{\omega} a_k\boldsymbol{x}_k+a_i\left(\boldsymbol{x}_i+\xi_n\boldsymbol{y}_j\right)+a_j\left(\boldsymbol{x}_j+\xi_n\alpha_{ij}\boldsymbol{y}_i\right),$$

where $\boldsymbol{z}'\in V^{\perp}$ and $a_k\in K$ for all $1\leq k\leq\omega$. Then we have

$$\begin{aligned} F(\boldsymbol{z}) &= a_ia_jF\left(\boldsymbol{x}_i+\xi_n\boldsymbol{y}_j,\boldsymbol{x}_j+\xi_n\alpha_{ij}\boldsymbol{y}_i\right)\\ &= a_ia_j\xi_n\left(\alpha_{ij}F(\boldsymbol{x}_i,\boldsymbol{y}_i)+F(\boldsymbol{x}_j,\boldsymbol{y}_j)\right)=0. \end{aligned}$$

Therefore W^n_{ij} is a maximal totally isotropic subspace of (V,F) for each pair $(i,j)\in\mathcal{I}_{\omega J}$ and $n\geq 1$. Also, if $J>0$, then for any integers $n,m\geq 1$ and ordered pairs $(i,j),(i',j')\in\mathcal{I}_{\omega J}$,

$$\dim_K\left(W^n_{ij}\cap W^m_{i'j'}\right)=\begin{cases}\omega+\lambda-1 & \text{if } n\neq m \text{ and } (i,j)=(i',j')\\ \omega+\lambda-3 & \text{if } (i,j)\neq(i',j').\end{cases}\tag{66}$$

If, on the other hand, $J=0$, then

$$\dim_K\left(W^n_{ij}\cap W^m_{i'j'}\right)=\begin{cases}\omega+\lambda-2 & \text{if } n\neq m \text{ and } (i,j)=(i',j')\\ \omega+\lambda-2 & \text{if } (i,j)=(j',i')\\ \omega+\lambda-4 & \text{if none of } i,j \text{ equals any of } i',j'.\end{cases}\tag{67}$$

In particular, notice that for each ordered triple (n,i,j) we get a different maximal totally isotropic subspace W^n_{ij} of (V,F). Moreover, for each $n\geq 1$, define

$$W^n:=\operatorname{span}_K\{W^n_{ij}:(i,j)\in\mathcal{I}_{\omega J}\}\subseteq V,$$

then $V^{\perp},\boldsymbol{x}_1,\dots,\boldsymbol{x}_{\omega}\in W^n$. If, in addition $J>0$, then all of

$$\xi_n^2\alpha_{11}\boldsymbol{y}_1+\xi_n\boldsymbol{u}_1,\dots,\xi_n^2\alpha_{1J}\boldsymbol{y}_1+\xi_n\boldsymbol{u}_J,\dots,\xi_n^2\alpha_{\omega 1}\boldsymbol{y}_{\omega}+\xi_n\boldsymbol{u}_1,\dots,\xi_n^2\alpha_{\omega 1}\boldsymbol{y}_{\omega}+\xi_n\boldsymbol{u}_J$$

are also in W^n. If, on the other hand, $J=0$, then $\boldsymbol{y}_1,\dots,\boldsymbol{y}_{\omega}\in W^n$. Therefore it is easy to see that in both cases W^n contains $\lambda+2\omega+J=L$ linearly independent vectors, hence $W^n=V$ for each $n\geq 1$.

Finally, we want to estimate the height of W^n_{ij} for $(i,j)\in\mathcal{I}_{\omega J}$ and $n\geq 1$. First assume $J>0$. By Lemma 2.2,

$$\mathcal{H}(W^n_{ij})\leq N^{\delta\omega/2}\mathcal{H}(V^{\perp})H(\boldsymbol{x}_i+\xi_n^2\alpha_{ij}\boldsymbol{y}_i+\xi_n\boldsymbol{u}_j)\prod_{k=1,k\neq i}^{\omega}H(\boldsymbol{x}_k),\tag{68}$$

where δ is as in (28). Now, by Lemma 2.1,

$$H(\boldsymbol{x}_i+\xi_n^2\alpha_{ij}\boldsymbol{y}_i+\xi_n\boldsymbol{u}_j)\leq 3^{\delta}H(1,\xi_n,\xi_n^2\alpha_{ij})h(\boldsymbol{x}_i)h(\boldsymbol{y}_i)h(\boldsymbol{u}_j).\tag{69}$$

Notice that by (62),

$$\begin{aligned} H(1,\xi_n,\xi_n^2\alpha_{ij}) &= H\left(2F(\boldsymbol{x}_i,\boldsymbol{y}_i),2\xi_nF(\boldsymbol{x}_i,\boldsymbol{y}_i),\xi_n^2F(\boldsymbol{u}_j)\right)\\ &\leq 2^{\delta}h(1,\xi_n^2)H\left(F(\boldsymbol{x}_i,\boldsymbol{y}_i),F(\boldsymbol{u}_j)\right)\\ &= 2^{\delta}h(1,\xi_n^2)H\left(\sum_{s=1}^{N}\sum_{t=1}^{N}f_{st}x_{is}y_{it},\sum_{s=1}^{N}\sum_{t=1}^{N}f_{st}u_{js}u_{jt}\right)\\ &\leq (2N^2)^{\delta}h(1,\xi_n^2)h(\boldsymbol{x}_i)h(\boldsymbol{y}_i)h(\boldsymbol{u}_j)^2H(F). \end{aligned}\tag{70}$$

Combining (68), (69), and (70) with (61), we obtain:

$$\begin{aligned}\mathcal{H}(W_{ij}^n) &\leq (6N^{\frac{\omega+4}{2}})^{\delta}(C_K(J)\mathcal{E}_K(J)^{1-\delta})^3 \times \\ &\quad \times H(F)\mathcal{H}(U)^3\mathcal{H}(V^{\perp})h(1,\xi_n^2)h(\boldsymbol{y}_i)^2 h(\boldsymbol{x}_i)\prod_{k=1}^{\omega} h(\boldsymbol{x}_k). \qquad (71)\end{aligned}$$

Now (71) combined with Lemma 4.2 implies that

$$\mathcal{H}(W_{ij}^n) \leq \mathfrak{C}'_K(N,\omega,J)a_K(n)H(F)^{\frac{\omega+9}{2}}\mathcal{H}(U)^3\mathcal{H}(V^{\perp})\mathcal{H}(\mathbb{H}_i)^7\prod_{k=1}^{\omega}\mathcal{H}(\mathbb{H}_k), \qquad (72)$$

where $a_K(n)$ is as in (4) if K is a number field or a function field of finite type, and

$$\mathcal{H}(W_{ij}^n) \leq \mathfrak{C}'_K(N,\omega,J)H(F)^{\frac{\omega+9}{2}}\mathcal{H}(U)^3\mathcal{H}(V^{\perp})\mathcal{H}(\mathbb{H}_i)^{10}\prod_{k=1}^{\omega}\mathcal{H}(\mathbb{H}_k)^2 \qquad (73)$$

if $K=\overline{\mathbb{Q}}$, where

$$\mathfrak{C}'_K(N,\omega,J) = \begin{cases} 9\times 2^{\frac{3\omega+19}{2}}N^{\frac{\omega+12}{2}}C_K(J)^3\mathcal{A}_K^4 B_K(1)^{2\omega+6} & \text{if } K \text{ n.f.} \\ 16q^{\frac{4(\omega+3)g(K)}{d}}\left(C_K(J)\mathcal{E}_K(J)\right)^3\mathcal{A}_K^4 & \text{if } K \text{ f.f.f.} \\ 1492992\times 72^{\omega+1}N^{\frac{\omega+12}{2}}C_K(J)^3\mathcal{A}_K^4 & \text{if } K=\overline{\mathbb{Q}}, \end{cases} \qquad (74)$$

and $\mathcal{A}_K = \mathcal{E}_K(2)A_K(2)C_K(2)$, as in Lemma 4.2. Now we combine (72), (73) with the bounds of Theorem 4.1 and Corollary 3.11 to obtain (2), where the exponents $\mathrm{p}_1(\omega)$, $\mathrm{p}_2(k,\omega)$, and $\mathrm{p}_3(k,\omega)$ are as in (3) and the field constant $\mathfrak{C}_K = \mathfrak{C}_K(N,\omega,J,L,r)$ is defined by

$$\mathfrak{C}_K = \begin{cases} \mathfrak{C}'_K(N,\omega,J)\mathcal{G}_K(N,L,\omega)^{\omega+10}a_K(n)B_K(r)^r & \text{if } K \text{ n.f.} \\ \mathfrak{C}'_K(N,\omega,J)\mathcal{G}_K(N,L,\omega)^{\omega+10}a_K(n)q^{\frac{rg(K)}{d}} & \text{if } K \text{ f.f.f.} \end{cases} \qquad (75)$$

when K is a number field or a function field of finite type, and

$$\mathfrak{C}_K = 8(2k+1)^{\frac{3}{2}}\mathfrak{C}'_K(N,\omega,J)\eta(L,r)^{\mathrm{p}_3(k,\omega)}3^{\frac{(10k+11)(k+2)k+48k^4\,\mathrm{p}_2(k,\omega)}{2(k+2)}}k^{\frac{\mathrm{p}_2(k,\omega)}{2}} \qquad (76)$$

when $K=\overline{\mathbb{Q}}$.

Now assume $J=0$, then employing the same kind of estimates as above we obtain:

$$\begin{aligned}\mathcal{H}(W_{ij}^n) &\leq N^{\delta\omega/2}\mathcal{H}(V^{\perp})H(\boldsymbol{x}_i+\xi_n\boldsymbol{y}_j)H(\boldsymbol{x}_j+\xi_n\alpha_{ij}\boldsymbol{y}_i)\prod_{k=1,k\neq i,j}^{\omega}H(\boldsymbol{x}_k) \\ &\leq N^{\delta\omega/2}\mathcal{H}(V^{\perp})H(1,\xi_n)H(1,\xi_n\alpha_{ij})h(\boldsymbol{y}_i)h(\boldsymbol{y}_j)\prod_{k=1}^{\omega}h(\boldsymbol{x}_k) \\ &\leq N^{\delta\omega/2}\mathcal{H}(V^{\perp})H(1,\xi_n)^2 H\left(F(\boldsymbol{x}_i,\boldsymbol{y}_i),F(\boldsymbol{x}_j,\boldsymbol{y}_j)\right)h(\boldsymbol{y}_i)h(\boldsymbol{y}_j)\prod_{k=1}^{\omega}h(\boldsymbol{x}_k) \\ &\leq N^{\delta\omega/2}\mathcal{H}(V^{\perp})H(1,\xi_n)^2 H(F)h(\boldsymbol{y}_i)^2 h(\boldsymbol{y}_j)^2 h(\boldsymbol{x}_i)h(\boldsymbol{x}_j)\prod_{k=1}^{\omega}h(\boldsymbol{x}_k). \qquad (77)\end{aligned}$$

Then again Lemma 4.2 together with Theorem 4.1 and Corollary 3.11 imply (5) where the exponents $\mathrm{q}_1(\omega)$ and $\mathrm{q}_2(k,\omega)$ are as in (6) and the constant $\mathfrak{F}_K =$

$\mathfrak{F}_K(N,\omega,L,r)$ is defined by

(78)
$$\mathfrak{F}_K = \begin{cases} 81 \times 2^{\frac{3\omega}{2}+17} N^{\frac{\omega+16}{2}} B_K(1)^{2\omega+12} B_K(r)^r \mathcal{A}_K^8 \mathcal{G}_K(N,L,\omega)^{\omega+14} & \text{if } K \text{ n.f.} \\ 256 \times q^{\frac{g(K)(4\omega+r+24)}{d}} \mathcal{A}_K^8 \mathcal{G}_K(N,L,\omega)^{\omega+14} & \text{if } K \text{ f.f.f.} \end{cases}$$

when K is a number field or a function field of finite type, and

$$\mathfrak{F}_K = 2^{3\omega+26} 3^{\frac{24k^4\,\mathfrak{q}_2(k,\omega)}{k+2}+\frac{L(L-1)}{2}+2\omega+16} N^{\frac{\omega+16}{2}}\ \mathcal{A}_K^8 k^{\frac{\mathfrak{q}_2(k,\omega)}{2}} \eta(L,r)^{\frac{(6k+5)\,\mathfrak{q}_2(k,\omega)}{4k+2}} \tag{79}$$

when $K = \overline{\mathbb{Q}}$. This completes the proof of the theorem. □

5. Isotropic points missing varieties

In this section we prove Theorem 1.3 and Corollary 1.4. We start with a non-vaninishing lemma for polynomials, which is a generalization of Theorem 3.1 of [8] over a variety of fields with a height function. It is an immediate corollary of an argument in [11].

Lemma 5.1. *Let K be a number field, function field, or algebraic closure of one or the other. Let $N, M \geq 1$ be integers and let $P(\boldsymbol{X}) := P(X_1,\dots,X_N) \in K[X_1,\dots,X_N]$ be a polynomial that is not identically zero with* $\deg P \leq M$*. Then there exists $\boldsymbol{z} \in K^N$ such that $P(\boldsymbol{z}) \neq 0$ and*

$$h(\boldsymbol{z}) \leq A_K(M),$$

where $A_K(M)$ is as in (20).

Proof. The conclusion of the lemma follows immediately from Lemma 4.1 of [11] combined with the argument in Section 7 of [11] (in particular, see formulas (44) and (45) of [11]). □

We will also need a technical lemma providing a bound on the height of a restriction of a polynomial to a subspace.

Lemma 5.2. *Let K be a number field, function field, or algebraic closure of one or the other. Let $N, M \geq 1$ be integers and let $P(\boldsymbol{X}) := P(X_1,\dots,X_N) \in K[X_1,\dots,X_N]$ be a polynomial of degree M. Let $V \subseteq K^N$ be an L-dimensional subspace, $1 \leq L \leq N$, such that P is not identically zero on V. Let $\boldsymbol{x}_1,\dots,\boldsymbol{x}_L$ be a basis for V over K, write A for the $N \times L$ basis matrix $(\boldsymbol{x}_1,\dots,\boldsymbol{x}_L)$, and define*

$$P_A(Y_1,\dots,Y_L) = P(Y_1\boldsymbol{x}_1 + \dots + Y_L\boldsymbol{x}_L) \in K[Y_1,\dots,Y_L],$$

so that P_A is a restriction of P to V. Then P_A is a polynomial of degree M in L variables over K, and

$$H(P_A) \leq L^{\delta M} H(P) \prod_{i=1}^{L} h(\boldsymbol{x}_i)^M, \tag{80}$$

where δ is as in (28).

Proof. Notice that

$$P_A(Y_1,\dots,Y_L) = P\left(\sum_{i=1}^{L} x_{i1}Y_i, \dots, \sum_{i=1}^{L} x_{iL}Y_i\right),$$

and so for each $v \mid \infty$,

$$H_v(P_A) \leq L^M H_v(P) \max_{1 \leq i \leq L, 1 \leq j \leq N} |x_{ij}|_v^{Md_v} \leq L^M H_v(P) \prod_{i=1}^{L} H_v(1, \boldsymbol{x}_i)^M,$$

while for each $v \nmid \infty$,

$$H_v(P_A) \leq H_v(P) \max_{1 \leq i \leq L, 1 \leq j \leq N} |x_{ij}|_v^{Md_v} \leq H_v(P) \prod_{i=1}^{L} H_v(1, \boldsymbol{x}_i)^M.$$

Then (80) follows by taking a product over all places of K while keeping in mind that function fields have no archimedean places. It should be remarked that in case $K = \overline{K}$, the product is taken over all places of the smallest subfield of K containing coefficients of P and coordinates of $\boldsymbol{x}_1, \dots, \boldsymbol{x}_L$, which is consistent with our definition of heights over $\overline{K}$. □

Our next lemma establishes a basic divisibility property of a polynomial with respect to any fixed monomial ordering.

Lemma 5.3. *Let K be any field, and let $P_1(\boldsymbol{X}), P_2(\boldsymbol{X}) \in K[\boldsymbol{X}] := K[X_1, \dots, X_N]$ be two polynomials in $N \geq 1$ variables over K. Fix any monomial ordering. Then there exist polynomials $P_1'(\boldsymbol{X}), R(\boldsymbol{X}) \in K[\boldsymbol{X}]$ such that*

$$P_1(\boldsymbol{X}) = P_1'(\boldsymbol{X}) + R(\boldsymbol{X}) P_2(\boldsymbol{X}), \tag{81}$$

and the leading monomial of $P_2(\boldsymbol{X})$, with respect to our chosen monomial ordering, does not divide any monomial of $P_1'(\boldsymbol{X})$.

Proof. Let us write $\mathfrak{L}(P_2)$ for the leading monomial of $P_2(\boldsymbol{X})$ with respect to the chosen monomial order. If $\mathfrak{L}(P_2)$ does not divide any monomial of P_1, then set $P_1'(\boldsymbol{X}) = P_1(\boldsymbol{X})$ and $R(\boldsymbol{X}) = 0$, and (81) follows. Hence assume that $\mathfrak{L}(P_2)$ divides at least one monomial of $P_1(\boldsymbol{X})$, and among all such monomials let $c_{\boldsymbol{a}_0} \boldsymbol{X}^{\boldsymbol{a}_0} := c_{\boldsymbol{a}_0} X_1^{a_{01}} \dots X_N^{a_{0N}}$ be the leading one with respect to our chosen monomial order, where $\boldsymbol{a}_0 = (a_{01}, \dots, a_{0N}) \in \mathbb{Z}_{\geq 0}$ and $c_{\boldsymbol{a}_0} \in K$. Define

$$g_1(\boldsymbol{X}) = P_1(\boldsymbol{X}) - \frac{c_{\boldsymbol{a}_0} \boldsymbol{X}^{\boldsymbol{a}_0}}{\mathfrak{L}(P_2)} P_2(\boldsymbol{X}).$$

Now for each $i \geq 1$, let $\boldsymbol{a}_i \in \mathbb{Z}_{\geq 0}$ be such that $c_{\boldsymbol{a}_i} \boldsymbol{X}^{\boldsymbol{a}_i}$ be the leading monomial of $g_i(\boldsymbol{X})$ divisible by $\mathfrak{L}(P_2)$. If such monomial exists, define

$$g_{i+1}(\boldsymbol{X}) = g_i(\boldsymbol{X}) - \frac{c_{\boldsymbol{a}_i} \boldsymbol{X}^{\boldsymbol{a}_i}}{\mathfrak{L}(P_2)} P_2(\boldsymbol{X}).$$

Notice that the set of vectors $\boldsymbol{a}_i$ as above forms a discrete subset of $\mathbb{R}^N$, which is decreasing with respect to the L_1-norm

$$|\boldsymbol{z}|_1 := |z_1| + \dots + |z_N|,$$

and is bounded from below by $\boldsymbol{0}$, hence it must be a finite set. This means that the process we described terminates, and so there exists some positive integer k such that no monomial of $g_k(\boldsymbol{x})$ is divisible by $\mathfrak{L}(P_2)$. Therefore

$$P_1(\boldsymbol{X}) = g_k(\boldsymbol{X}) + \sum_{i=1}^{k-1} \frac{c_{\boldsymbol{a}_i} \boldsymbol{X}^{\boldsymbol{a}_i}}{\mathfrak{L}(P_2)} P_2(\boldsymbol{X}),$$

and so (81) holds with

$$P_1'(\boldsymbol{X}) = g_k(\boldsymbol{X}),\ R(\boldsymbol{X}) = \sum_{i=1}^{k-1} \frac{c_{\boldsymbol{a}_i} \boldsymbol{X}^{\boldsymbol{a}_i}}{\mathfrak{L}(P_2)},$$

both having the required properties. □

We always write $\boldsymbol{X}$ for the variable vector $(X_1, \dots, X_N)$. Let $I \subsetneq \{1, \dots, N\}$, and write $\boldsymbol{X}'_I$ for the vector of all variables in $\boldsymbol{X}$ whose indices are not in I. In the same way, given a vector $\boldsymbol{z} \in K^N$, we will write $\boldsymbol{z}'_I$ for the vector one obtains from $\boldsymbol{z}$ by removing the coordinates indexed by I; we also write $K_I^{N-|I|}$ for the space of all such vectors over the field K. Finally, let us write d_P for the degree of a polynomial P. The next lemma establishes the existence of zeros of especially small height for polynomials of arbitrary degree away from a hypersurface, provided the polynomial in question is of a particular form.

Lemma 5.4. *Let K be a number field, function field, or algebraic closure of one or the other, and let $N \geq 3$ be an integer. Let $Q(\boldsymbol{X}) \in K[\boldsymbol{X}]$ be a polynomial of the form*

$$Q(\boldsymbol{X}) = X_i X_j (c + Q_1(\boldsymbol{X}'_{\{i,j\}})) + Q_2(\boldsymbol{X}'_{\{i,j\}}) \tag{82}$$

for some indices $1 \leq i < j \leq N$, where $0 \neq c \in K$ and Q_1, Q_2 are polynomials in the $N-2$ variables $\boldsymbol{X}'_{\{i,j\}}$. Assume that Q has non-trivial zeros over K, and let $P(\boldsymbol{X}) \in K[\boldsymbol{X}]$ be a polynomial such that there exists $\boldsymbol{0} \neq \boldsymbol{z} \in K^N$ with $Q(\boldsymbol{z}) = 0$ and $P(\boldsymbol{z}) \neq 0$. Then there exists such $\boldsymbol{z}$ with

$$H(\boldsymbol{z}) \leq h(\boldsymbol{z}) \leq A_K(d_P + d_Q)^{d_Q+1} A_K(2d_P)^2 H(Q), \tag{83}$$

where $A_K(M)$ is the constant given by (20) above.

Proof. Let us choose a monomial ordering with respect to which the leading monomial of $Q(\boldsymbol{X})$ contains the product X_iX_j. Then Lemma 5.3 guarantees the existence of polynomials $P'(\boldsymbol{X}), R(\boldsymbol{X}) \in K[X_1, \dots, X_N]$ such that $P = P' + RQ$ and X_iX_j does not divide $P'(\boldsymbol{X})$. Since for any $\boldsymbol{z} \in K^N$ with $Q(\boldsymbol{z}) = 0$, $P(\boldsymbol{z}) = P'(\boldsymbol{z})$, we can assume from the start that X_iX_j does not divide $P(\boldsymbol{X})$, replacing P with P' if necessary. Then we can write $P(\boldsymbol{X})$ in the form

$$P(\boldsymbol{X}) = X_i^k g_1(\boldsymbol{X}'_{\{j\}}) + g_2(\boldsymbol{X}'_{\{i\}}),$$

for some positive integer k and $(N-1)$-variable polynomials $g_1(\boldsymbol{X}'_{\{j\}})$, $g_2(\boldsymbol{X}'_{\{i\}})$, where $g_1(\boldsymbol{X}'_{\{j\}})$ is either identically zero or has a monomial not divisible by X_i.

First assume that $g_1 = 0$, then X_i does not divide any monomial of P, meaning that $P(\boldsymbol{X}) = g_2(\boldsymbol{X}'_{\{i\}}) \neq 0$ is a polynomial in the $N-1$ variables $\boldsymbol{X}'_{\{i\}}$. Then Lemma 5.1 implies that there exists $\boldsymbol{z}'_{\{i\}} \in K_{\{i\}}^{N-1}$ such that

$$P(\boldsymbol{z}) z_j (c + Q_1(\boldsymbol{z}'_{\{i,j\}})) = g_2(\boldsymbol{z}'_{\{i\}}) z_j (c + Q_1(\boldsymbol{z}'_{\{i,j\}})) \neq 0,$$

for any $z_i \in K$, and

$$h(\boldsymbol{z}'_{\{i\}}) \leq A_K(d_P + d_{Q_1} + 1) \leq A_K(d_P + d_Q). \tag{84}$$

Then, for this choice of $\boldsymbol{z}'_{\{i\}}$, let

$$z_i = -\frac{Q_2(\boldsymbol{z}'_{\{i,j\}})}{z_j (c + Q_1(\boldsymbol{z}'_{\{i,j\}}))},$$

and notice that $Q(\boldsymbol{z})=0$, $P(\boldsymbol{z})\neq 0$. Also notice that

$$\begin{aligned} h(\boldsymbol{z}) &\leq H(1,z_i)h(\boldsymbol{z}'_{\{i\}}) \leq A_K(d_P+d_Q)H(Q_2(\boldsymbol{z}'_{\{i,j\}}), z_j(c+Q_1(\boldsymbol{z}'_{\{i,j\}})))\\ &\leq A_K(d_P+d_Q)h(\boldsymbol{z}'_{\{i\}})^{d_Q}H(Q) \leq A_K(d_P+d_Q)^{d_Q+1}H(Q). \end{aligned} \tag{85}$$

Next suppose that $g_1 \neq 0$. Define $r(\boldsymbol{X}'_{\{i,j\}})$ to be the sum of all monomials of $g_1(\boldsymbol{X}'_{\{j\}})$ which are not divisible by X_i. Since $g_1\neq 0$, it must have a monomial not divisible by X_i, and so $r\neq 0$. Similarly to the argument above, there must exist $\boldsymbol{z}'_{\{i,j\}} \in K^{N-2}_{\{i\}}$ such that $r(\boldsymbol{z}'_{\{i,j\}})\neq 0$ for any $z_i,z_j\in K$, and

$$h(\boldsymbol{z}'_{\{i,j\}}) \leq A_K(d_r) \leq A_K(d_{g_1}) \leq A_K(d_P-k). \tag{86}$$

Now define

$$g_1'(X_i) := g_1(z_1,\dots,z_{i-1},X_i,z_{i+1},\dots,z_{j-1},z_{j+1},\dots,z_N),$$

and

$$g_2'(X_j) := g_2(z_1,\dots,z_{i-1},z_{i+1},\dots,z_{j-1},X_j,z_{j+1},\dots,z_N).$$

Notice in particular that $g_1'(X_i)$ is not identically zero, since its nonzero constant term is $r(\boldsymbol{z}'_{\{i,j\}})$. For each $X_i\neq 0$, define a vector function of X_i:

$$\boldsymbol{f}_{\boldsymbol{z}'_{\{i,j\}}}(X_i) = \left(z_1,\dots,z_{i-1},X_i,z_{i+1},\dots,z_{j-1},-\frac{Q_2(\boldsymbol{z}'_{\{i,j\}})}{X_i(c+Q_1(\boldsymbol{z}'_{\{i,j\}}))},z_{j+1},\dots,z_N\right).$$

Then $Q(\boldsymbol{f}_{\boldsymbol{z}'_{\{i,j\}}}(X_i))=0$ for any $X_i\neq 0$. Moreover, $P(\boldsymbol{f}_{\boldsymbol{z}'_{\{i,j\}}}(X_i))\neq 0$ for any nonzero value of X_i for which the polynomial

$$\begin{aligned} P'(X_i) &:= X_i^{d_{g_2'}}P(\boldsymbol{f}_{\boldsymbol{z}'_{\{i,j\}}}(X_i))\\ &= X_i^{k+d_{g_2'}}g_1'(X_i) + X_i^{d_{g_2'}}g_2'\left(-\frac{Q_2(\boldsymbol{z}'_{\{i,j\}})}{X_i(c+Q_1(\boldsymbol{z}'_{\{i,j\}}))}\right) \end{aligned}$$

is nonzero. Notice that $X_i^{k+d_{g_2'}}g_1'(X_i)$ is a nonzero polynomial of degree

$$k+d_{g_1'}+d_{g_2'} > d_{g_2'},$$

since $k>0$, and $X_i^{d_{g_2'}}g_2'\left(-\frac{Q_2(\boldsymbol{z}'_{\{i,j\}})}{X_i(c+Q_1(\boldsymbol{z}'_{\{i,j\}}))}\right)$ is a polynomial of degree $d_{g_2'}$. Therefore $P'(X_i)$ is not identically zero, and hence Lemma 5.1 implies that there exists $0\neq\alpha\in K$ such that $P'(\alpha)\neq 0$ and

$$h(\alpha) \leq A_K(d_{P'}) \leq A_K(k+d_{g_1'}+d_{g_2'}) \leq A_K(2d_P). \tag{87}$$

Now take $\boldsymbol{z} = \boldsymbol{f}_{\boldsymbol{z}'_{\{i,j\}}}(\alpha)$, then we have $Q(\boldsymbol{z})=0$, $P(\boldsymbol{z})\neq 0$, and

$$\begin{aligned} h(\boldsymbol{z}) &= H\left(1,\boldsymbol{z}'_{\{i,j\}},\alpha,\frac{Q_2(\boldsymbol{z}'_{\{i,j\}})}{\alpha(c+Q_1(\boldsymbol{z}'_{\{i,j\}}))}\right)\\ &\leq h(\boldsymbol{z}'_{\{i,j\}})h(\alpha)H\left(1,\frac{Q_2(\boldsymbol{z}'_{\{i,j\}})}{\alpha(c+Q_1(\boldsymbol{z}'_{\{i,j\}}))}\right)\\ &= h(\boldsymbol{z}'_{\{i,j\}})h(\alpha)H\left(Q_2(\boldsymbol{z}'_{\{i,j\}}),\alpha(c+Q_1(\boldsymbol{z}'_{\{i,j\}}))\right)\\ &\leq h(\boldsymbol{z}'_{\{i,j\}})^{d_Q+1}h(\alpha)^2H(Q) \leq A_K(d_P-k)^{d_Q+1}A_K(2d_P)^2H(Q), \end{aligned} \tag{88}$$

where the last inequality follows by combining (86) and (87).

Now (83) follows by combining (85) and (88), and this finishes the proof of the lemma. □

We are now ready for the main argument of this section.

Proposition 5.5. *Let K be a number field (n.f.), function field of finite type (f.f.f.), or $\overline{\mathbb{Q}}$, $N \geq 2$ an integer, and F a quadratic form in N variables over K. Let $V \subseteq K^N$ be an L-dimensional subspace, $1 \leq L \leq N$, and suppose that the quadratic space (V, F) has rank $1 \leq r \leq L$ and λ is the dimension of $V^{\perp}$, the radical of V. Let $P(\boldsymbol{X}) \in K[X_1, \dots, X_N]$ be a polynomial of degree M, and suppose that there exists $\boldsymbol{0} \neq \boldsymbol{z} \in V$ such that $F(\boldsymbol{z}) = 0$ and $P(\boldsymbol{z}) \neq 0$. Then there exists such a point $\boldsymbol{z} \in V$ with*

$$(89) \quad H(\boldsymbol{z}) \leq h(\boldsymbol{z}) \leq \begin{cases} T_K(L,M) H(F)^{\frac{9L+11}{2}} \mathcal{H}(V)^{9L+12} & \text{if } K \text{ n.f. or f.f.f.} \\ T_K(L,M) H(F)^{\max\{r, 29/2\}} \mathcal{H}(V)^{30} & \text{if } K = \overline{\mathbb{Q}}, \end{cases}$$

where

$$(90) \quad T_K(L,M) = 3^3 2^{\frac{21L-21}{2}} L^{\frac{27L+51}{2}} C_K(L)^{9L+14} B_K(L-1)^9 A_K(M+2)^3 A_K(2M)^2 |\mathcal{D}_K|^{\frac{9}{2d}},$$

when K is a number field,

$$(91) \quad T_K(L,M) = q^{\frac{(18L^2-27L+18)g(K)}{d}} C_K(L)^{9L+15} \mathcal{E}_K(L)^{9L+15} A_K(M+2)^3 A_K(2M)^2,$$

when K is a function field of finite type, and

$$(92) \quad T_K(L,M) = 3^{18(L-\lambda)+\frac{18L(L-1)}{L-\lambda}+3} L^{51} C_K(L)^{33} A_K(1)^3 A_K(M+2)^3 A_K(2M)^2,$$

when $K = \overline{\mathbb{Q}}$.

Proof. First suppose that P is not identically zero on $V^{\perp}$, then Theorem 1.4 of [11] implies that there exists $\boldsymbol{0} \neq \boldsymbol{z} \in V^{\perp}$ such that $P(\boldsymbol{z}) \neq 0$ and

$$H(\boldsymbol{z}) \leq h(\boldsymbol{z}) \leq \lambda A_K(M) C_K(\lambda) \mathcal{H}(V^{\perp}).$$

Combining this observation with (34) and (50) above, we obtain:

$$(93) \quad H(\boldsymbol{z}) \leq h(\boldsymbol{z}) \leq \begin{cases} B_K(r)^r \lambda A_K(M) C_K(\lambda) H(F)^{r/2} \mathcal{H}(V) & \text{if } K \text{ n.f.} \\ q^{\frac{rg(K)}{d}} \lambda A_K(M) C_K(\lambda) H(F)^{r/2} \mathcal{H}(V) & \text{if } K \text{ f.f.f.} \\ 3^{\frac{L(L-1)}{2}} \lambda A_K(M) C_K(\lambda) H(F)^r \mathcal{H}(V)^2 & \text{if } K = \overline{\mathbb{Q}}, \end{cases}$$

and since $F(\boldsymbol{z}) = 0$, we are done.

Next assume that P is identically zero on $V^{\perp}$. Then there must exist some nonsingular zero of F on V at which P does not vanish; in particular, F must have nonsingular zeros on V, so if, say, $L = 1$, then we must have $V = \operatorname{span}_K\{\boldsymbol{x}\}$ where $F(\boldsymbol{x}) = 0$, $P(\boldsymbol{x}) \neq 0$ (clearly, $H(\boldsymbol{x}) = \mathcal{H}(V)$ in this case), and if $L = 2$, then V must be a hyperbolic plane. Let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_L$ be the small-height basis for V, guaranteed by Siegel's lemma (see [1] and [17] for original results, and Theorems 1.1, 1.2 of [11] for a convenient formulation):

$$(94) \quad \prod_{i=1}^{L} h(\boldsymbol{x}_i) \leq C_K(L) \mathcal{E}_K(L)^{1-\delta} \mathcal{H}(V).$$

Let $A = (\boldsymbol{x}_1 \dots \boldsymbol{x}_L)$ be the corresponding basis matrix and let F_A, P_A be the corresponding restrictions of F and P to V as defined in Lemma 5.2. Combining (80) with (94), we obtain

$$H(F_A) \leq \left(L^{\delta} C_K(L) \mathcal{E}_K(L)^{1-\delta}\right)^2 H(F) \mathcal{H}(V)^2. \tag{95}$$

Now notice that for each $\boldsymbol{z} \in K^L$, $F_A(\boldsymbol{z}) = 0$, $P_A(\boldsymbol{z}) = 0$ if and only if $F(A\boldsymbol{z}) = 0$, $P(A\boldsymbol{z}) = 0$, respectively. Moreover, $\boldsymbol{z} \in K^L$ is a nonsingular zero of F_A if and only if $A\boldsymbol{z} \in V$ is a non-singular zero of F. Also notice that by Lemma 2.1 combined with (94)

$$h(A\boldsymbol{z}) = h\left(\sum_{i=1}^{L} z_i \boldsymbol{x}_i\right) \leq L^{\delta} h(\boldsymbol{z}) \prod_{i=1}^{L} h(\boldsymbol{x}_i) \leq L^{\delta} C_K(L) \mathcal{E}_K(L)^{1-\delta} h(\boldsymbol{z}) \mathcal{H}(V). \tag{96}$$

Since P does not vanish at some nonsingular zero of F on V, it must be that P_A does not vanish at some nonsingular zero of F on K^L; in particular, the quadratic space (K^L, F_A) must contain a hyperbolic plane. Our next task will be to find a hyperbolic pair of bounded height in (K^L, F_A).

Corollary 1.2 of [8], in case K is a number field, and Corollary 3.8 above, in case K is function field of finite type, guarantee the existence of a nonsingular zero of $\boldsymbol{x} \in K^L$ of F_A with

$$\begin{aligned} h(\boldsymbol{x}) &\leq \begin{cases} 2^{\frac{3(L-1)}{2}} L^{\frac{L-1}{2}} |\mathcal{D}_K|^{\frac{1}{2d}} B_K(L-1) H(F_A)^{\frac{L-1}{2}} & \text{if } K \text{ n.f.} \\ q^{\frac{(2L^2-3L+2)g(K)}{d}} H(F_A)^{\frac{L-1}{2}} & \text{if } K \text{ f.f.f.} \end{cases} \\ &\leq \begin{cases} |\mathcal{D}_K|^{\frac{1}{2d}} B_K(L-1) \left((2L)^{\frac{3}{2}} C_K(L) H(F)^{\frac{1}{2}} \mathcal{H}(V)\right)^{L-1} & \text{if } K \text{ n.f.} \\ q^{\frac{(2L^2-3L+2)g(K)}{d}} \left(C_K(L) \mathcal{E}_K(L) H(F)^{\frac{1}{2}} \mathcal{H}(V)\right)^{L-1} & \text{if } K \text{ f.f.f.,} \end{cases} \end{aligned} \tag{97}$$

where the last inequality follows by (95). If $K = \overline{\mathbb{Q}}$, then Lemma 3.5 of [10] states that the quadratic space (K^L, F_A) can be represented as $K^L = \left(K^L\right)^{\perp} \perp W$, where W is a regular subspace of K^L and $\mathcal{H}(W) \leq 3^{\frac{L(L-1)}{2}}$. Since the quadratic spaces (K^L, F_A) and (V, F) are isometric, their radicals have the same dimension. Therefore, the dimensions of $(K^L)^{\perp}$ and W are λ and $L - \lambda$, respectively. Then Lemma 4.1 of [10] states that there exists $\boldsymbol{0} \neq \boldsymbol{x} \in W$ (hence $\boldsymbol{x}$ is a nonsingular point in (K^L, F_A)) with

$$\begin{aligned} h(\boldsymbol{x}) &\leq 8 \times 3^{2(L-\lambda-1)} H(F_A)^{\frac{1}{2}} \mathcal{H}(W)^{\frac{4}{L-\lambda}} \\ &\leq 3^{2(L-\lambda) + \frac{2L(L-1)}{L-\lambda}} L C_K(L) H(F)^{\frac{1}{2}} \mathcal{H}(V), \end{aligned} \tag{98}$$

where the last inequality follows by (95).

Now let K be a number field, function field, or $\overline{\mathbb{Q}}$, and let $\boldsymbol{x}$ be a nonsingular point satisfying (97) or (98), respectively. Since $\boldsymbol{x}$ is nonsingular, the linear form $F_A(\boldsymbol{x}, \boldsymbol{Y})$ is not identically zero on K^L, and so there must exist a standard basis vector in K^L, call it $\boldsymbol{u}$, such that $F_A(\boldsymbol{x}, \boldsymbol{u}) \neq 0$ and $h(\boldsymbol{u}) = 1$. Then $\mathbb{H}_{xu} := \operatorname{span}_K\{\boldsymbol{x}, \boldsymbol{u}\}$ is a hyperbolic plane in (K^L, F_A) with

(99) $\mathcal{H}(\mathbb{H}_{xu}) \leq L^{\delta} H(\boldsymbol{x}) H(\boldsymbol{u})$

$$\leq \begin{cases} (2L)^{\frac{3L-3}{2}} L|\mathcal{D}_K|^{\frac{1}{2d}} B_K(L-1) C_K(L)^{L-1} H(F)^{\frac{L-1}{2}} \mathcal{H}(V)^{L-1} & \text{if } K \text{ n.f.} \\ q^{\frac{(2L^2-3L+2)g(K)}{d}} \left(C_K(L)\mathcal{E}_K(L) H(F)^{\frac{1}{2}} \mathcal{H}(V) \right)^{L-1} & \text{if } K \text{ f.f.f.} \\ 3^{2(L-\lambda)+\frac{2L(L-1)}{L-\lambda}} L^2 C_K(L) H(F)^{\frac{1}{2}} \mathcal{H}(V) & \text{if } K = \overline{\mathbb{Q}}, \end{cases}$$

where the first inequality is given by Lemma 2.2 and the second follows by (97) and (98). Let also

$$\boldsymbol{y} = F_A(\boldsymbol{u})\boldsymbol{x} - 2F_A(\boldsymbol{x}, \boldsymbol{u})\boldsymbol{u},$$

then $F_A(\boldsymbol{y}) = 0$ and $F_A(\boldsymbol{x}, \boldsymbol{y}) \neq 0$, so $\boldsymbol{x}, \boldsymbol{y}$ is a hyperbolic pair. Moreover, (58) and (59) state that

$$h(\boldsymbol{y}) \leq \left(3L^2\right)^{\delta} H(F_A) h(\boldsymbol{x}) h(\boldsymbol{u})^2 = \left(3L^2\right)^{\delta} H(F_A) h(\boldsymbol{x}).$$

Combining this observation with (97), (98), and (95). we obtain that

(100)

$$h(\boldsymbol{y}) \leq \begin{cases} 3 \times 2^{\frac{3(L-1)}{2}} L^{\frac{3L+5}{2}} |\mathcal{D}_K|^{\frac{1}{2d}} B_K(L-1) \left(C_K(L) H(F)^{\frac{1}{2}} \mathcal{H}(V) \right)^{L+1} & K \text{ n.f.} \\ q^{\frac{(2L^2-3L+2)g(K)}{d}} \left(C_K(L)\mathcal{E}_K(L) H(F)^{\frac{1}{2}} \mathcal{H}(V) \right)^{L+1} & K \text{ f.f.f.} \end{cases}$$

when K is a number field or a function field of finite type, and

$$h(\boldsymbol{y}) \leq 3^{2(L-\lambda)+\frac{2L(L-1)}{L-\lambda}+1} L^5 C_K(L)^3 H(F)^{\frac{3}{2}} \mathcal{H}(V)^3, \tag{101}$$

when $K = \overline{\mathbb{Q}}$.

Define

$$\begin{aligned} \mathbb{H}'_{xu} &:= \left\{ \boldsymbol{v} \in K^L : F_A(\boldsymbol{v}, \boldsymbol{z}) = 0 \ \forall \ \boldsymbol{z} \in \mathbb{H}_{xu} \right\} \\ &= \left\{ \boldsymbol{v} \in K^L : (\boldsymbol{x}\ \boldsymbol{u})^t F_A \boldsymbol{v} = 0 \ \forall \ \boldsymbol{z} \in \mathbb{H}_{xu} \right\} \end{aligned}$$

to be the $(L-2)$-dimensional orthogonal complement of $\mathbb{H}_{xu}$ in (K^L, F_A); here we also write F_A for the coefficient matrix of the quadratic form F_A. By the Brill-Gordan duality principle discussed in Section 2 above, $\mathcal{H}(\mathbb{H}'_{xu})$ is precisely the vector space height $\mathcal{H}$ of the matrix $(\boldsymbol{x}\ \boldsymbol{u})^t F_A$, and hence Lemma 2.3 implies that

$$\mathcal{H}(\mathbb{H}'_{xu}) \leq L^{3\delta} H(F_A)^2 H(\boldsymbol{x}) H(\boldsymbol{u}),$$

and then (95) combined with (99) imply that

(102)

$$\mathcal{H}(\mathbb{H}'_{xu}) \leq \begin{cases} 2^{\frac{3L-3}{2}} L^{\frac{3L+11}{2}} |\mathcal{D}_K|^{\frac{1}{2d}} B_K(L-1) \left(C_K(L) H(F)^{\frac{1}{2}} \mathcal{H}(V) \right)^{L+3} & K \text{ n.f.} \\ q^{\frac{(2L^2-3L+2)g(K)}{d}} \left(C_K(L)\mathcal{E}_K(L) H(F)^{\frac{1}{2}} \mathcal{H}(V) \right)^{L+3} & K \text{ f.f.f.} \end{cases}$$

when K is a number field or a function field of finite type, and

$$\mathcal{H}(\mathbb{H}'_{xu}) \leq 3^{2(L-\lambda)+\frac{2L(L-1)}{L-\lambda}} L^9 A_K(1) C_K(L)^5 H(F)^{\frac{5}{2}} \mathcal{H}(V)^5, \tag{103}$$

when $K = \overline{\mathbb{Q}}$. Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{L-2}$ be the small-height basis for $\mathbb{H}'_{xu}$, guaranteed by Siegel's lemma:

$$\prod_{i=1}^{L-2} h(\boldsymbol{v}_i) \leq C_K(L-2)\mathcal{E}_K(L-2)^{1-\delta} \mathcal{H}(\mathbb{H}'_{xu}) \leq C_K(L)\mathcal{E}_K(L)^{1-\delta} \mathcal{H}(\mathbb{H}'_{xu}). \tag{104}$$

Combining (104) with (102) and (103), we see that

(105)
$$\prod_{i=1}^{L-2} h(\boldsymbol{v}_i) \leq \begin{cases} 2^{\frac{3L-3}{2}} L^{\frac{3L+11}{2}} |\mathcal{D}_K|^{\frac{1}{2d}} B_K(L-1) C_K(L)^{L+4} \left(H(F)^{\frac{1}{2}} \mathcal{H}(V)\right)^{L+3} & K \text{ n.f.} \\ q^{\frac{(2L^2-3L+2)g(K)}{d}} \left(C_K(L)\mathcal{E}_K(L)\right)^{L+4} \left(H(F)^{\frac{1}{2}} \mathcal{H}(V)\right)^{L+3} & K \text{ f.f.f.} \end{cases}$$

when K is a number field or a function field of finite type, and

$$\prod_{i=1}^{L-2} h(\boldsymbol{v}_i) \leq 3^{2(L-\lambda)+\frac{2L(L-1)}{L-\lambda}} L^9 A_K(1) C_K(L)^6 H(F)^{\frac{5}{2}} \mathcal{H}(V)^5, \tag{106}$$

when $K = \overline{\mathbb{Q}}$. Now define the matrix $B = (\boldsymbol{x}\ \boldsymbol{y}\ \boldsymbol{v}_1 \dots \boldsymbol{v}_{L-2}) \in \mathrm{GL}_L(K)$, and let

$$G(\boldsymbol{Y}) = F_A(B\boldsymbol{Y}),\ Q(\boldsymbol{Y}) = P_A(B\boldsymbol{Y}).$$

Then it is easy to see that G is of the form (82), and so G and Q satisfy the conditions of Lemma 5.4. Hence Lemma 5.4 guarantees the existence of a point $\boldsymbol{w} \in K^L$ such that $G(\boldsymbol{w}) = 0$, $Q(\boldsymbol{w}) \neq 0$, and

$$h(\boldsymbol{w}) \leq A_K(M+2)^3 A_K(2M)^2 H(G). \tag{107}$$

Now notice that standard height inequalities along with (95) imply that

$$\begin{aligned} H(G) &\leq H(B^t F_A B) \leq L^{2\delta} H(B)^2 H(F_A) \leq L^{2\delta} H(F_A) h(\boldsymbol{x})^2 h(\boldsymbol{y})^2 \prod_{i=1}^{L-2} h(\boldsymbol{v}_i)^2 \\ &\leq L^{4\delta} C_K(L)^2 \mathcal{E}_K(L)^{2(1-\delta)} H(F) \mathcal{H}(V)^2 h(\boldsymbol{x})^2 h(\boldsymbol{y})^2 \prod_{i=1}^{L-2} h(\boldsymbol{v}_i)^2, \end{aligned}$$

and so by combining (105), (106) with (97), (98), (100), and (101), we see that $H(G)$ is bounded above by

(108)
$$\begin{cases} 3^2 2^{6L-6} L^{9L+17} |\mathcal{D}_K|^{\frac{3}{d}} B_K(L-1)^6 C_K(L)^{6L+9} \left(H(F)\mathcal{H}(V)^2\right)^{3L+4} & K \text{ n.f.} \\ q^{\frac{(12L^2-18L+12)g(K)}{d}} C_K(L)^{6L+10} \mathcal{E}_K(L)^{6L+10} \left(H(F)\mathcal{H}(V)^2\right)^{3L+4} & K \text{ f.f.f.} \end{cases}$$

when K is a number field or a function field of finite type, and

$$H(G) \leq 3^{12(L-\lambda)+\frac{12L(L-1)}{L-\lambda}+2} L^{34} A_K(1)^2 C_K(L)^{22} H(F)^{10} \mathcal{H}(V)^{20}, \tag{109}$$

when $K = \overline{\mathbb{Q}}$. Now define $\boldsymbol{z} = A(B\boldsymbol{w}) \in V$, and notice that $F(\boldsymbol{z}) = F_A(B\boldsymbol{w}) = G(\boldsymbol{z}) = 0$, and $P(\boldsymbol{z}) = P_A(B\boldsymbol{w}) = Q(\boldsymbol{w}) \neq 0$. Hence $\boldsymbol{z}$ is precisely the point we are looking for, and to estimate its height first notice that by the same kind of reasoning as in (96),

$$h(B\boldsymbol{w}) = h\left(w_1\boldsymbol{x} + w_2\boldsymbol{y} + \sum_{i=1}^{L-2} w_{i+2}\boldsymbol{v}_i\right) \leq L^{\delta} h(\boldsymbol{w}) h(\boldsymbol{x}) h(\boldsymbol{y}) \prod_{i=1}^{L} h(\boldsymbol{v}_i). \tag{110}$$

Then combining (110) with (96), (107), (108), (97), (100), and (105) we obtain (89) in the number field or function field case, which is larger than the corresponding bound in (93). On the other hand, when $K = \overline{\mathbb{Q}}$, we can combine (110) with (96), (107), (109), (98), (101), and (106) to obtain

$$h(\boldsymbol{z}) \leq T_K(L,M) H(F)^{10} \mathcal{H}(V)^{21}, \tag{111}$$

where $T_K(L,M)$ is as in (92). Combining (111) with the corresponding bound of (93), we obtain (89) in the $\overline{\mathbb{Q}}$ case. This completes the proof of the proposition. □

Remark 5.1. Notice that it is also easy to obtain a version of Lemma 5.4 with a restriction to a subspace V of K^N instead of the whole K^N by applying Lemma 5.2 in the same way as we do it in the proof of Proposition 5.5.

Proof of Theorem 1.3. Let the notation be as in the statement of Theorem 1.3. We start by extending the result of Proposition 5.5 to a statement about a small-height zero of F in V outside of the union of varieties $\mathcal{Z}_K$ as defined in (7). Since $Z(V,F) \nsubseteq \mathcal{Z}_K$, $Z(V,F) \nsubseteq Z_K(P_{i1},\dots,P_{ik_i})$ for all $1 \leq i \leq J$, and so for each i at least one of the polynomials $P_{i1},\dots,P_{ik_i}$ is not identically zero on $Z(V,F)$, say it is P_{ij_i} for some $1 \leq j_i \leq k_i$. Clearly for each $1 \leq i \leq J$, $Z_K(P_{i1},\dots,P_{ik_i}) \subseteq Z_K(P_{ij_i})$, and $\deg(P_{ij_i}) = m_{ij_i} \leq M_i$. Define

$$P(X_1,\dots,X_N) = \prod_{i=1}^{J} P_{ij_i}(X_1,\dots,X_N),$$

so that $Z(V,F) \nsubseteq Z_K(P)$ while $\mathcal{Z}_K \subseteq Z_K(P)$. Then it is sufficient to construct a point of bounded height $\boldsymbol{x} \in Z(V,F) \setminus Z_K(P)$. Now notice that $\deg(P) = \sum_{i=1}^{J} m_{ij_i} \leq M$ and apply Proposition 5.5.

Next we want to prove the existence of a linearly independent collection of vectors $\boldsymbol{x}_1,\dots,\boldsymbol{x}_m \in Z(V,F) \setminus \mathcal{Z}_K$ satisfying (9) and (10), where $m = \omega + \lambda$. Proposition 5.5, along with the argument above, guarantee the existence of a point $\boldsymbol{x}_1 \in Z(V,F) \setminus \mathcal{Z}_K$ satisfying (89). In fact, let $\boldsymbol{x}_1$ be a point of smallest height possible in $Z(V,F) \setminus \mathcal{Z}_K$ satisfying (89). If $m = 1$, we are done; hence suppose that $m > 1$. Then there must exist a maximal totally isotropic subspace W_1 of (V,F) containing $\boldsymbol{x}_1$, and so $W_1 \nsubseteq \mathcal{Z}_K$ and $\dim_K W_1 = m$. Then, as implied by Theorem A.1, W_1 has a full basis $\boldsymbol{u}_1,\dots,\boldsymbol{u}_m$ outside of $\mathcal{Z}_K$. Let $\mathfrak{X}_1$ be an $(N-1)$-dimensional subspace of K^N containing $\boldsymbol{x}_1$ so that $W_1 \nsubseteq \mathfrak{X}_1$, then at least one of $\boldsymbol{u}_1,\dots,\boldsymbol{u}_m$ is not in $\mathfrak{X}_1$. Since $W_1 \subseteq Z(V,F)$, we can conclude that $Z(V,F) \nsubseteq \mathcal{Z}_K^1 := \mathcal{Z}_K \cup \mathfrak{X}_1$, and $M(\mathcal{Z}_K^1) = M(\mathcal{Z}_K) + 1$, since $\mathfrak{X}_1$ is the nullspace of a linear form. Again, Proposition 5.5, along with the argument above, guarantee the existence of a point $\boldsymbol{x}_2 \in Z(V,F) \setminus \mathcal{Z}_K^1$ satisfying (89) with $M = M(\mathcal{Z}_K)+1$, and we can assume that $\boldsymbol{x}_2$ is a point of smallest height possible in $Z(V,F) \setminus \mathcal{Z}_K^1$ satisfying (89). If $m = 2$, we are done; then assume $m > 2$. Then there must exist a maximal totally isotropic subspace W_2 of (V,F) containing $\boldsymbol{x}_1, \boldsymbol{x}_2$, and so $W_2 \nsubseteq \mathcal{Z}_K$ and $\dim_K W_2 = m$. Again, Theorem A.1 guarantees that W_2 has a full basis $\boldsymbol{u}_1',\dots,\boldsymbol{u}_m'$ outside of $\mathcal{Z}_K$. Let $\mathfrak{X}_2$ be an $(N-1)$-dimensional subspace of V containing vectors $\boldsymbol{x}_1, \boldsymbol{x}_2$, and let $\mathcal{Z}_K^2 = \mathcal{Z}_K \cup \mathfrak{X}_2$. Then $V \nsubseteq \mathcal{Z}_K^2$ and $M(\mathcal{Z}_K^2) = M(\mathcal{Z}_K) + 1$. Continuing to apply Proposition 5.5 and Theorem 1.4 of [11] in the same manner, we construct a collection of linearly independent vectors $\boldsymbol{x}_1,\dots,\boldsymbol{x}_m \in V \setminus \mathcal{Z}_K$ satisfying (116) and (117). This completes the proof of the theorem. □

We can now prove Corollary 1.4.

Proof of Corollary 1.4. We first show that for each $1 \leq n \leq m = \omega+\lambda$, there exists a maximal totally isotropic subspace W_n^m of (V,F) of bounded height, containing the corresponding point $\boldsymbol{x}_n$ from the statement of Theorem 1.3; since $\boldsymbol{x}_n \notin \mathcal{Z}_K$, it follows that $W_n^m \nsubseteq \mathcal{Z}_K$. First suppose that $\boldsymbol{x}_n \in V^{\perp}$, then $\mathcal{Z}_K$ cannot contain any maximal totally isotropic subspace of (V,F), since each one of them contains $V^{\perp}$. Hence we can pick W_n^m to be a maximal totally isotropic subspace of (V,F) of bounded height as guaranteed by Theorem 1 of [21] in case K is a number field, by Theorem 1.1 above in case K is a function field of finite type, and by Theorem 1.1 and Lemma 3.5 of [10] in case $K=\overline{\mathbb{Q}}$. Next assume that $\boldsymbol{x}_n$ is a nonsingular point, then define

$$U_n = \{\boldsymbol{z} \in V : F(\boldsymbol{z},\boldsymbol{x}_n)=0\} = \{\boldsymbol{z} \in K^N : \boldsymbol{z}^t(F\boldsymbol{x}_n)=0\} \cap V,$$

so that $\dim_K U_n = L-1$ and

$$\mathcal{H}(U_n) \leq \mathcal{H}(F\boldsymbol{x}_n)\mathcal{H}(V) \leq N^{3\delta/2} H(F) H(\boldsymbol{x}_n) \mathcal{H}(V), \tag{112}$$

by the Brill-Gordan duality principle (discussed in Section 2 above), combined with Lemmas 2.3 and 2.4 above. Let W_n' be a maximal totally isotropic subspace of (U_n,F) of bounded height as guaranteed by Theorem 1 of [21] in case K is a number field, by Theorem 1.1 above in case K is a function field of finite type, and by Theorem 1.1 and Lemma 3.5 of [10] in case $K=\overline{\mathbb{Q}}$. Therefore

$$\mathcal{H}(W_n') \leq \left(2^{2m+1} B_K (L-m-1)^2 H(F)\right)^{L-m-1} \mathcal{H}(U_n), \tag{113}$$

when K is a number field, and

$$\mathcal{H}(W_n') \leq q^{\frac{(L-m-1)^2 g(K)}{d}} H(F)^{L-m-1} \mathcal{H}(U_n), \tag{114}$$

when K is a function field of finite type. If $K=\overline{\mathbb{Q}}$, we obtain

$$\mathcal{H}(W_n') \leq 3^{2(\omega-1)\omega^3+\frac{(4\omega+1)(L-1)(L-2)}{6}} H(F)^{(\omega-1)^2+r} \mathcal{H}(U_n)^{\frac{4\omega+4}{3}}. \tag{115}$$

Here our bound is slightly worth than what follows from Theorem 1.1 and Lemma 3.5 of [10], however in this form it is easier to read and apply. Now define $W_n^m = \operatorname{span}_K\{\boldsymbol{x}_n, W_n'\}$, then W_n^m is a maximal totally isotropic subspace of (V,F) containing $\boldsymbol{x}_n$. Moreover,

$$\mathcal{H}(W_n^m) \leq N^{\delta/2}\ H(\boldsymbol{x}_n)\mathcal{H}(W_n').$$

Combining this observation with (113), (114), (115), and the bounds of Theorem 1.3, we obtain (11) in the case $k=m$.

Now Siegel's lemma implies the existence of a basis $\boldsymbol{w}_1,\dots,\boldsymbol{w}_m$ for W_n^m such that

$$\prod_{i=1}^m h(\boldsymbol{w}_i) \leq C_K(m)\mathcal{E}_K(m)^{1-\delta}\mathcal{H}(W_n^m).$$

Since $\boldsymbol{0} \neq \boldsymbol{x}_n \in W_n^m$, there must exist a subcollection of $m-1$ of these vectors which are linearly independent with $\boldsymbol{x}_n$; since we did not order these vectors by height, we can assume without loss of generality that $\boldsymbol{x}_n, \boldsymbol{w}_2, \dots, \boldsymbol{w}_m$ are linearly independent. Then for each $1 \leq k < m$, define

$$W_n^k = \operatorname{span}_K\{\boldsymbol{x}_n, \boldsymbol{w}_2, \dots, \boldsymbol{w}_k\},$$

so that $\boldsymbol{x}_n \in W_n^k$, $\dim_K W_n^k = k$,

$$\operatorname{span}_K\{\boldsymbol{x}_n\} = W_n^1 \subset W_n^2 \subset \dots \subset W_n^m,$$

and by Lemma 2.2

$$\mathcal{H}(W_n^k) \leq N^{\delta k/2} H(\boldsymbol{x}_n) \prod_{i=2}^{k} h(\boldsymbol{w}_i) \leq N^{\delta k/2} C_K(m) \mathcal{E}_K(m)^{1-\delta} H(\boldsymbol{x}_n) \mathcal{H}(W_n^m),$$

which is precisely (16). This completes the proof of the corollary. □

Appendix A. Linear bases of small height

In this appendix we present two different variations of Siegel's lemma over function field. First we use Theorem 1.4 of [11] to establish the following result on the existence of a full basis of small height for a linear space outside of a union of varieties. Unlike in the rest of the paper, the function fields here can have any perfect field for their coefficient fields, i.e., can be of finite or infinite type.

Theorem A.1. *Let K be a number field, $\overline{\mathbb{Q}}$, or the function field of a smooth projective curve over a perfect field. Let $N \geq 2$ be an integer, and let V be an L-dimensional subspace of K^N, $1 \leq L \leq N$. Let $\mathcal{Z}_K$ and $M = M(\mathcal{Z}_K)$ be as in* (7), (8) *above. Suppose that $V \nsubseteq \mathcal{Z}_K$. Then there exists a basis $\boldsymbol{x}_1, \dots, \boldsymbol{x}_L \in V \setminus \mathcal{Z}_K$ for V over K such that*

$$H(\boldsymbol{x}_1) \leq H(\boldsymbol{x}_2) \leq \dots \leq H(\boldsymbol{x}_L), \ h(\boldsymbol{x}_1) \leq h(\boldsymbol{x}_2) \leq \dots \leq h(\boldsymbol{x}_L), \tag{116}$$

and for each $1 \leq n \leq L$,

$$H(\boldsymbol{x}_n) \leq h(\boldsymbol{x}_n) \leq L^{\delta} \mathcal{E}_K(L)^{1-\delta} A_K(M+1) C_K(L) \mathcal{H}(V), \tag{117}$$

where δ is as in (28)*, $C_K(L)$ is as in (19), $A_K(M)$ is as in (20), and $\mathcal{E}_K(L)$ is as in (18).*

Proof. We want to prove the existence of a linearly independent collection of vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_L \in V \setminus \mathcal{Z}_K$ satisfying (116) and (117). Theorem 1.4 of [11] guarantees the existence of a point $\boldsymbol{x}_1 \in V \setminus \mathcal{Z}_K$ with

$$H(\boldsymbol{x}_1) \leq h(\boldsymbol{x}_1) \leq L^{\delta} \mathcal{E}_K(L)^{1-\delta} A_K(M) C_K(L) \mathcal{H}(V). \tag{118}$$

In fact, let $\boldsymbol{x}_1$ be a point of smallest height possible in $V \setminus \mathcal{Z}_K$ satisfying (118). If $L = 1$, we are done. If not, let $\mathfrak{X}_1$ be an $(N-1)$-dimensional subspace of K^N containing $\boldsymbol{x}_1$ and not containing the entire V, and let $\mathcal{Z}_K^1 = \mathcal{Z}_K \cup \mathfrak{X}_1$. Then $V \nsubseteq \mathcal{Z}_K^1$ and $M(\mathcal{Z}_K^1) = M(\mathcal{Z}_K) + 1$, since $\mathfrak{X}_1$ is the nullspace of a linear form. Again, Theorem 1.4 of [11] guarantees the existence of a point $\boldsymbol{x}_2 \in V \setminus \mathcal{Z}_K^1$ with

$$H(\boldsymbol{x}_2) \leq h(\boldsymbol{x}_2) \leq L^{\delta} \mathcal{E}_K(L)^{1-\delta} A_K(M+1) C_K(L) \mathcal{H}(V), \tag{119}$$

and we can assume that $\boldsymbol{x}_2$ is a point of smallest height possible in $V \setminus \mathcal{Z}_K^1$ satisfying (119). If $L = 2$, we are done. If not, we can let $\mathfrak{X}_2$ be an $(N-1)$-dimensional subspace of K^N containing vectors $\boldsymbol{x}_1, \boldsymbol{x}_2$ and not containing the entire V, and let $\mathcal{Z}_K^2 = \mathcal{Z}_K \cup \mathfrak{X}_2$. Then $V \nsubseteq \mathcal{Z}_K^2$ and $M(\mathcal{Z}_K^2) = M(\mathcal{Z}_K) + 1$. Continuing to apply Theorem 1.4 of [11] in the same manner, we construct a collection of linearly independent vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_L \in V \setminus \mathcal{Z}_K$ satisfying (116) and (117). This completes the proof of the theorem. □

Remark A.1. Notice that Theorem 1.3 follows from Proposition 5.5 and Theorem 1.4 of [11] by a similar kind of repetitive application argument as used in the proof of Theorem A.1.

Secondly, we state an "orthogonal" version of Siegel's lemma over a function field, which is a direct adaptation of Theorem 2.4 of [9] proved in the number field case.

Theorem A.2. *Let K be a function field and $V \subset K^N$ an L-dimensional space of K^N, $1 \leq L < N$. Let F be a quadratic form in N variable over K, where we also write F for the associated symmetric bilinear form. Then there exists a basis $\boldsymbol{x}_1, \dots, \boldsymbol{x}_L$ for V over K such that $F(\boldsymbol{x}_i, \boldsymbol{x}_j) = 0$ for all $i \neq j$, and*

$$\prod_{i=1}^{L} H(\boldsymbol{x}_i) \leq C_K(L)^{\frac{L^2+L-2}{4}} H(F)^{\frac{L(L+1)}{2}} H(V)^L,$$

where $C_K(L)$ is as in (19).

The proof of Theorem A.2 is identical to the proof of Theorem 2.4 of [9], where we use Thunder's function field version of Siegel's lemma instead of the Bombieri-Vaaler version over a number field.

Appendix B. Effective results on isometries of quadratic spaces

In this appendix, let K be a function field of finite or infinite type over a perfect constant field with $\operatorname{char}(K) \neq 2$. Here we discuss the effective structure of the isometry group of a bilinear space over K. For the rest of this appendix assume that $V \subseteq K^N$ is an L-dimensional subspace, $1 \leq L \leq N$ such that the bilinear space (V, F) is regular. Our arguments are completely parallel to their respective analogues over a number field and $\overline{\mathbb{Q}}$ (see [9], [10], respectively), while keeping in mind that all places of a function field are non-archimedean. We include them here for the purposes of self-containment and readability. Our main goal here is to prove a slightly weaker effective version of the classical Cartan-Dieudonné theorem, Theorem B.4 below. From here on our notation is the same as in Section 5 of [9]; we review it here.

First notice that $K^N = V \perp (\perp_{K^N}(V))$, where $\perp_{K^N}(V) = \{\boldsymbol{x} \in K^N : F(\boldsymbol{x}, \boldsymbol{z}) = 0 \ \forall \ \boldsymbol{z} \in V\}$, as above. Let $\mathcal{O}(V, F)$ be the group of isometries of (V, F), and write id_V for its identity element. Also let $-id_V$ be the element of $\mathcal{O}(V, F)$ that takes $\boldsymbol{x}$ to $-\boldsymbol{x}$ for each $\boldsymbol{x} \in V$. Each element σ of the isometry group $\mathcal{O}(K^N, F)$ is uniquely represented by an $N \times N$ matrix $A \in GL_N(K)$, and so we can define $H(\sigma)$ to be the height of A viewed as a vector in K^{N^2}, same way as for the coefficient matrix of F.

Notice that each $\sigma \in \mathcal{O}(V, F)$ can be extended to an isometry $\hat{\sigma} \in \mathcal{O}(K^N, F)$ by selecting an isometry $\sigma' \in \mathcal{O}(\perp_{K^N}(V), F)$. For each $\sigma \in \mathcal{O}(V, F)$ choose such an extension $\hat{\sigma} : K^N \to K^N$ by selecting σ' among those with determinant ± 1 which minimizes $H(\hat{\sigma})$, and define $H(\sigma) = H(\hat{\sigma})$ for this choice of $\hat{\sigma}$. This definition of height in particular ensures that for each $\sigma \in \mathcal{O}(V, F)$

$$H(\sigma) = H(-\sigma), \tag{120}$$

where $-\sigma = -id_V \circ \sigma$. Moreover, if A is the matrix of $\hat{\sigma}$, then

$$\det(A) = \det(\hat{\sigma}) = \det\left(\hat{\sigma}\mid_V\right) \det\left(\hat{\sigma}\mid_{\perp_{K^N}(V)}\right) = \det(\sigma)\det(\sigma') = \pm 1. \tag{121}$$

We will also refer to this matrix A as the matrix of σ.

For each $\boldsymbol{x} \in V$ such that $F(\boldsymbol{x}) \neq 0$ we can define an element of $\mathcal{O}(V,F)$, $\tau_{\boldsymbol{x}} : V \longrightarrow V$, given by

$$\tau_{\boldsymbol{x}}(\boldsymbol{y}) = \boldsymbol{y} - \frac{2F(\boldsymbol{x},\boldsymbol{y})}{F(\boldsymbol{x})}\boldsymbol{x}, \tag{122}$$

which is a *reflection* in the hyperplane $\{\boldsymbol{x}\}^{\perp} = \{\boldsymbol{z} \in V : F(\boldsymbol{x},\boldsymbol{z}) = 0\}$. It is not difficult to see that the matrix of such a reflection on the entire space K^N is of the form $(\tau_{ij}(\boldsymbol{x}))_{1\leq i,j\leq N}$, where

$$\tau_{ij}(\boldsymbol{x}) = \begin{cases} 1 - \frac{2}{F(\boldsymbol{x})}\sum_{k=1}^N f_{ik}x_ix_k & \text{if } i = j \\ -\frac{2}{F(\boldsymbol{x})}\sum_{k=1}^N f_{jk}x_ix_k & \text{if } i \neq j \end{cases}$$

For each reflection $\tau_{\boldsymbol{x}}$, $\det(\tau_{\boldsymbol{x}}) = -1$. We say that σ is a *rotation* if $\det(\sigma) = +1$.

We can now derive some bounds on height of isometries of (V,F). As in Section 3, we keep in mind that over a function field the heights $\mathcal{H}$ and H are the same, and so we use notation H for all homogeneous heights. We start with a simple result, which is precisely Lemma 5.1 of [9], adapted to account for non-archimedean places only.

Lemma B.1. *Let $\boldsymbol{x} \in V$ be anisotropic and $\tau_{\boldsymbol{x}} \in \mathcal{O}(V,F)$ be the corresponding reflection. Then*

$$H(\tau_{\boldsymbol{x}}) = H(\hat{\tau}_{\boldsymbol{x}}) \leq H\left((\tau_{ij}(\boldsymbol{x}))_{1\leq i,j\leq N}\right) \leq H(F)H(\boldsymbol{x})^2. \tag{123}$$

The next lemma is a direct analogue of Lemma 5.2 of [9].

Lemma B.2. *There exists an anisotropic vector $\boldsymbol{x}$ in V such that $\sigma(\boldsymbol{x}) \pm \boldsymbol{x}$ is also anisotropic for some choice of $\pm$, and*

$$H(\boldsymbol{x}) \leq h(\boldsymbol{x}) \leq 2\left(C_K(L+2)\mathcal{E}_K(L+2)\right)^{\frac{1}{2}} H(V)^{\frac{L+2}{2L}}. \tag{124}$$

Proof. The argument is identical to that in the proof of Lemma 5.2 of [9] with the Bombieri-Vaaler version of Siegel's lemma replaced with the inhomogeneous height function field version (see [11, Theorem 1.2]). □

An immediate consequence of Lemmas B.1 and B.2 is the following statement about the existence of a reflection of relatively small height in $\mathcal{O}(V,F)$ - this is a direct analogue of Corollary 5.3 of [9].

Corollary B.3. *There exists a reflection $\tau \in \mathcal{O}(V,F)$ with*

$$H(\tau) \leq 4C_K(L+2)\mathcal{E}_K(L+2)H(F)H(V)^{\frac{L+2}{L}}. \tag{125}$$

Proof. Let $\boldsymbol{x}$ be an anisotropic point in V guaranteed by Lemma B.2. Let $\tau = \tau_{\boldsymbol{x}}$. The result follows by combining (123) with (124). □

Moreover, every isometry $\sigma \in \mathcal{O}(V,F)$ can be represented as a product of reflections of bounded height. This is an effective version of the well-known Cartan-Dieudonné theorem. Specifically, we can state the following.

Theorem B.4. *Let (V,F) be a regular symmetric bilinear space over K with $V \subseteq K^N$ of dimension L, $1 \leq L \leq N$, $N \geq 2$. Let $\sigma \in \mathcal{O}(V,F)$. Then either σ is the identity, or there exist an integer $1 \leq l \leq 2L-1$ and reflections $\tau_1, ..., \tau_l \in \mathcal{O}(V,F)$ such that*

$$\sigma = \tau_1 \circ \cdots \circ \tau_l, \tag{126}$$

and for each $1 \leq i \leq l$,

$$H(\tau_i) \leq \left\{ \left(2\left(C_K(L+2)\mathcal{E}_K(L+2)\right)^{\frac{1}{2}} \right)^{L^2} H(F)^{\frac{L}{3}} H(V)^{\frac{L}{2}} H(\sigma) \right\}^{5^{L-1}}. \tag{127}$$

To prove Theorem B.4 we will need the following two technical lemmas, which are immediate adaptations of Lemmas 5.4 and 5.6 of [9], respectively.

Lemma B.5. *Let* $A \in GL_N(K)$ *be such that* $\det(A) = \pm 1$*, and write* I_N *for the* $N \times N$ *identity matrix. Then*

$$H(A \pm I_N) \leq H(A). \tag{128}$$

Lemma B.6. *Let* A *and* B *be two* $N \times N$ *matrices with entries in* K*. Then*

$$H(AB) \leq H(A)H(B). \tag{129}$$

Proof of Theorem B.4. We argue by induction on L. When $L = 1$, $V = K\boldsymbol{x}$ for some anisotropic vector $\boldsymbol{x} \in K^N$, since (V, F) is regular. Then $\sigma = \pm id_V$, where $-id_V = \tau_{\boldsymbol{x}}$, and $H(\sigma) = 1$ by (120).

Then assume $L > 1$. Write A for the $N \times N$ matrix of σ, and I_N for the $N \times N$ identity matrix, so in particular $H(\sigma) = H(A)$. Notice that for each $\boldsymbol{x} \in V$,

$$F(\sigma(\boldsymbol{x}) - \boldsymbol{x}, \sigma(\boldsymbol{x}) + \boldsymbol{x}) = 0. \tag{130}$$

Let $\boldsymbol{x} \in V$ be the anisotropic vector guaranteed by Lemma B.2 with $\sigma(\boldsymbol{x}) \pm \boldsymbol{x}$ also anisotropic. For this choice of $\pm$, $\tau_{\sigma(\boldsymbol{x}) \pm \boldsymbol{x}}$ fixes $\sigma(\boldsymbol{x}) \mp \boldsymbol{x}$ and maps $\sigma(\boldsymbol{x}) \pm \boldsymbol{x}$ to $-(\sigma(\boldsymbol{x}) \pm \boldsymbol{x})$. Then $2\sigma(\boldsymbol{x}) = (\sigma(\boldsymbol{x})) + (\sigma(\boldsymbol{x}) - \boldsymbol{x})$ will be mapped to $(\sigma(\boldsymbol{x}) \mp \boldsymbol{x}) - (\sigma(\boldsymbol{x}) \pm \boldsymbol{x}) = \mp 2\boldsymbol{x}$. We can therefore observe that if $\sigma(\boldsymbol{x}) - \boldsymbol{x}$ is anisotropic, then

$$\sigma' = \tau_{\sigma(\boldsymbol{x}) - \boldsymbol{x}} \circ \sigma \tag{131}$$

fixes $\boldsymbol{x}$. If, on the other hand, $\sigma(\boldsymbol{x}) + \boldsymbol{x}$ is anisotropic, then

$$\sigma' = \tau_{\sigma(\boldsymbol{x}) + \boldsymbol{x}} \circ \tau_{\sigma(\boldsymbol{x})} \circ \sigma \tag{132}$$

fixes $\boldsymbol{x}$. In any case, σ' defined either by (131) or (132) is an isometry of the $(L-1)$-dimensional regular bilinear space $(\{\boldsymbol{x}\}^{\perp}, F)$, where $\{\boldsymbol{x}\}^{\perp} = \{\boldsymbol{z} \in V : F(\boldsymbol{x}, \boldsymbol{z}) = 0\}$. Then, by the induction hypothesis,

$$\sigma' = \tau_1 \circ \cdots \circ \tau_l,$$

for some reflections $\tau_1, ..., \tau_l$ with $1 \leq l \leq 2L - 3$ and

$$H(\tau_i) \leq \left\{ \left(2\left(C_K(L+1)\mathcal{E}_K(L+1)\right)^{\frac{1}{2}} \right)^{(L-1)^2} H(F)^{\frac{L-1}{3}} H\left(\{\boldsymbol{x}\}^{\perp}\right)^{\frac{L-1}{2}} H(\sigma') \right\}^{5^{L-2}}, \tag{133}$$

for each $1 \leq i \leq l$, and so

$$\sigma = \sigma'' \circ \tau_1 \circ \cdots \circ \tau_l, \tag{134}$$

for the same $\tau_1, ..., \tau_l$ and $\sigma'' = \tau_{\sigma(\boldsymbol{x}) - \boldsymbol{x}}$ or $\sigma'' = \tau_{\sigma(\boldsymbol{x}) + \boldsymbol{x}} \circ \tau_{\sigma(\boldsymbol{x})}$, depending on which of $\sigma(\boldsymbol{x}) \pm \boldsymbol{x}$ is anisotropic, so σ is a product of at most $2L - 1$ reflections. Next we are going to produce bounds on their heights. Combining Lemma B.1 with a bound analogous to that of Lemma 2.3 and with Lemma B.2, we obtain

$$H(\tau_{\sigma(\boldsymbol{x})}) \leq 4C_K(L+2)\mathcal{E}_K(L+2)H(F)H(\sigma)^2 H(V)^{\frac{L+2}{L}}. \tag{135}$$

Therefore $\tau_{\sigma(\boldsymbol{x})}$ satisfies (127). Also by Lemma B.1,

$$H(\tau_{\sigma(\boldsymbol{x})\pm\boldsymbol{x}}) \leq H(F)H(\sigma(\boldsymbol{x})\pm\boldsymbol{x})^2. \tag{136}$$

Notice that $\sigma(\boldsymbol{x}) \pm \boldsymbol{x} = (A \pm I_N)\boldsymbol{x}$. Then, once again, by a bound analogous to that of Lemma 2.3

$$H(\sigma(\boldsymbol{x})\pm\boldsymbol{x}) \leq H(\boldsymbol{x})H(A\pm I_N) \leq 2\left(C_K(L+2)\mathcal{E}_K(L+2)\right)^{\frac{1}{2}} H(V)^{\frac{L+2}{2L}} H(A\pm I_N), \tag{137}$$

where the last inequality follows by (124). Combining (137) with Lemma B.5, we obtain

$$H(\sigma(\boldsymbol{x})\pm\boldsymbol{x}) \leq 2\left(C_K(L+2)\mathcal{E}_K(L+2)\right)^{\frac{1}{2}} H(V)^{\frac{L+2}{2L}} H(A). \tag{138}$$

Combining (136) and (138), we obtain

$$H(\tau_{\sigma(\boldsymbol{x})\pm\boldsymbol{x}}) \leq 4C_K(L+2)\mathcal{E}_K(L+2)H(F)H(V)^{\frac{L+2}{L}} H(\sigma)^2, \tag{139}$$

hence $\tau_{\sigma(\boldsymbol{x})\pm\boldsymbol{x}}$ satisfies (127). By combining (131), (132), (120), Lemma B.6, (135), and (139), we have

$$H(\sigma') \leq 16C_K(L+2)^2\mathcal{E}_K(L+2)^2H(F)^2H(V)^{\frac{2L+4}{L}} H(\sigma)^5. \tag{140}$$

By Lemma 2.4, Lemma 2.3, and (124)

$$H\left(\{\boldsymbol{x}\}^{\perp}\right) \leq H(F)H(\boldsymbol{x})H(V) \leq 2\left(C_K(L+2)\mathcal{E}_K(L+2)\right)^{\frac{1}{2}} H(F)H(V)^{\frac{3L+2}{2L}}. \tag{141}$$

Then bound (127) follows upon combining (133) with (140) and (141) while keeping in mind that $1 \leq L \leq N$ and that the constant $C_K(L)\mathcal{E}_K(L)$ is increasing with L. This completes the proof. □

Finally, we record a simple corollary of Lemma B.5, which provides a bound on the height of the invariant subspace of an isometry, an object of interest in the algebraic theory of quadratic forms. This is an immediate adaptation of Corollary 5.5 of [9].

Corollary B.7. *Let $\sigma \in \mathcal{O}(V,F)$. Let U be the invariant subspace of σ, i.e. $U = \{\boldsymbol{z} \in V : \sigma(\boldsymbol{z}) = \boldsymbol{z}\}$. Let $J = \dim_K(U) \leq L$. Then*

$$H(U) \leq H(\sigma)^{N-J}H(V). \tag{142}$$

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