ON AN EFFECTIVE VARIATION OF KRONECKER'S APPROXIMATION THEOREM AVOIDING ALGEBRAIC SETS

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Abstract. Let $\Lambda \subset \mathbb{R}^n$ be an algebraic lattice, coming from a projective module over the ring of integers of a number field K. Let $\mathcal{Z} \subset \mathbb{R}^n$ be the zero locus of a finite collection of polynomials such that $\Lambda \nsubseteq \mathcal{Z}$ or a finite union of proper full-rank sublattices of Λ. Let K_1 be the number field generated over K by coordinates of vectors in Λ, and let $L_1, \dots, L_t$ be linear forms in n variables with algebraic coefficients satisfying an appropriate linear independence condition over K_1. For each $\varepsilon > 0$ and $\boldsymbol{a} \in \mathbb{R}^n$, we prove the existence of a vector $\boldsymbol{x} \in \Lambda \setminus \mathcal{Z}$ of explicitly bounded sup-norm such that

$$\|L_i(\boldsymbol{x}) - a_i\| < \varepsilon$$

for each $1 \leq i \leq t$, where $\| \ \|$ stands for the distance to the nearest integer. The bound on sup-norm of $\boldsymbol{x}$ depends on ε, as well as on Λ, K, $\mathcal{Z}$ and heights of linear forms. This presents a generalization of Kronecker's approximation theorem, establishing an effective result on density of the image of $\Lambda \setminus \mathcal{Z}$ under the linear forms $L_1, \dots, L_t$ in the t-torus $\mathbb{R}^t/\mathbb{Z}^t$. In the appendix, we also discuss a construction of badly approximable matrices, a subject closely related to our proof of effective Kronecker's theorem, via Liouville-type inequalities and algebraic transference principles.

1. Introduction

Let $1, \theta_1, \dots, \theta_t$ be $\mathbb{Q}$-linearly independent real numbers. The classical approximation theorem of Kronecker then states that the set of points

$$\{(\{n\theta_1\}, \dots, \{n\theta_t\}) : n \in \mathbb{Z}\}$$

is dense in the t-torus $\mathbb{R}^t/\mathbb{Z}^t$, where $\{\cdot\}$ stands for the fractional part of a real number. This result was originally obtained by Kronecker [24] in 1884, and presents a deep generalization of Dirichlet's 1842 theorem on Diophantine approximation [6]; see, for instance, [20] for a detailed exposition of these classical results.

Kronecker's theorem can also be viewed as a statement on density of the image of the integer lattice under collection of linear forms in the torus $\mathbb{R}^t/\mathbb{Z}^t$ (compare to the famous Oppenheim conjecture for quadratic forms). Specifically, if $L_1, \dots, L_t$ are linear forms in n variables with real coefficients b_{ij} so that the set of numbers 1 and b_{ij} are linearly independent over $\mathbb{Q}$, then for any $\varepsilon > 0$ and $\boldsymbol{a} \in \mathbb{R}^t$ there exists $\boldsymbol{x} \in \mathbb{Z}^n$ such that

$$\|L_i(\boldsymbol{x}) - a_i\| < \varepsilon \ \forall \ 1 \leq i \leq t, \tag{1}$$

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where $\| \ \|$ stands for the distance to the nearest integer. A nice survey of a wide variety of results related to Kronecker's theorem is given in [18]. Classical quantitative results in this direction are related to transference theorems for homogeneous and inhomogeneous approximation for the system of linear forms $L_i(\boldsymbol{x})$ (see [22], Chapter V of [3], [2]). In particular, these results give effective bounds for the size of the coordinates of the vector $\boldsymbol{x}$ in (1) under the assumption that there are effective lower bounds for $\max_i \|L_i(\boldsymbol{x})\|$ in the homogeneous case. Some additional effective results can also be found in [26], [31].

The main goal of this note is somewhat different. We consider linear forms with algebraic coefficients and extend the previously known versions of Kronecker's theorem in three ways:

(1) allow for the approximating vector $\boldsymbol{x}$ as in the equation (1) above to come from an algebraic lattice Λ,
(2) exclude vectors from a prescribed union $\mathcal{Z}$ of projective varieties or sublattices not containing this lattice, that is we are interested in approximation vectors $\boldsymbol{x} \in \Lambda \setminus \mathcal{Z}$,
(3) we obtain effective constants everywhere in our upper bounds.

Effective Diophantine avoidance results, exhibiting solutions to a given problem outside of a prescribed algebraic set can be viewed as statements on distribution of such solutions: not only do small solutions exist, they are also sufficiently well distributed so that it is not possible to "cut them out" by any finite union of varieties. In the recent years, such results were obtained in the general context of Siegel's lemma (also generalizing Faltings' version of Siegel's lemma [8], [23], [7]) in [10], [11], [12], [15], [17], [21], and in the context of Cassels' theorem on small zeros of quadratic forms and its generalizations in [9], [5], [14], [16]. We will extend these investigations to Kronecker's theorem. To obtain effective constants in our bounds we use Liouville-type inequalities (see Remark 3.1 below for stronger non-effective inequalities of similar type, which can be derived from Schmidt's Subspace Theorem). To give precise statements of our results, we need some notation.

1. The lattice. Let $n \geq 1$ be an integer, and for each vector $\boldsymbol{x} \in \mathbb{R}^n$ define the sup-norm

$$|\boldsymbol{x}| := \max_{1 \leq i \leq n} |x_i|.$$

Let K be a number field of degree $d = r_1 + 2r_2$ over $\mathbb{Q}$, where r_1 and r_2 are numbers of its real and complex places, respectively, and write $\mathcal{O}_K$ for its ring of integers. Let $1 \leq s \leq w$ be integers, and let $\mathcal{M} \subset K^w$ be an $\mathcal{O}_K$-module such that $\mathcal{M} \otimes_K K \cong K^s$. Write $\mathcal{D}_K(\mathcal{M})$ for the discriminant of $\mathcal{M}$. Define $\mathfrak{U}_K(\mathcal{M})$, a fractional $\mathcal{O}_K$-ideal in K, to be

$$\mathfrak{U}_K(\mathcal{M}) = \{\alpha \in K : \alpha\mathcal{M} \subseteq \mathcal{O}_K^w\}. \tag{2}$$

We let $\Lambda_K(\mathcal{M}) \subset \mathbb{R}^{wd}$ be the lattice of rank sd, which is the image of $\mathcal{M}$ under the standard Minkowski embedding.

2. The projective varieties. Let $m \geq 1$ be an integer. For each $1 \leq i \leq m$, let $\mathcal{S}_i$ be a finite set of homogeneous polynomials in $\mathbb{R}[x_1, \dots, x_{wd}]$ and $Z(\mathcal{S}_i)$ be its zero set in $\mathbb{R}^{wd}$, that is,

$$Z(\mathcal{S}_i) = \{\boldsymbol{x} \in \mathbb{R}^{wd} : P(\boldsymbol{x}) = 0 \text{ for all } P \in \mathcal{S}_i\}.$$

For the collection $\boldsymbol{S} := \{\mathcal{S}_1, \dots, \mathcal{S}_m\}$ of finite sets of homogeneous polynomials, define

$$\mathcal{Z}_{\boldsymbol{S}} := \bigcup_{i=1}^{m} Z(\mathcal{S}_i), \tag{3}$$

and

$$M_{\boldsymbol{S}} := \sum_{i=1}^{m} \max\{\deg P : P \in \mathcal{S}_i\}. \tag{4}$$

We allow for the possibility that $\mathcal{Z}_{\boldsymbol{S}} = \{\mathbf{0}\}$, in which case we take instead $M_{\boldsymbol{S}} = 1$. Notice that $\mathcal{Z}_{\boldsymbol{S}}$ is an algebraic set, which is a union of a finite collection of projective varieties. Assume that the lattice $\Lambda_K(\mathcal{M})$ is not contained in the set $\mathcal{Z}_{\boldsymbol{S}}$.

3. The linear forms. Let $K_1 = K(\Lambda_K(\mathcal{M}))$, i.e. K_1 is the number field generated over K by the entries of any basis matrix of the lattice $\Lambda_K(\mathcal{M})$. Let $B := (b_{ij})_{1\leq i\leq t, 1\leq j\leq wd}$ be a $t \times wd$ matrix with real algebraic entries so that $1, b_{11}, \dots, b_{t(wd)}$ are linearly independent over K_1, and let $\ell = [E : \mathbb{Q}]$ where $E = K_1(b_{11}, \dots, b_{t(wd)})$. We will also write $\ell_v = [E_v : \mathbb{Q}_v]$ for the local degree of E at every place $v \in M(E)$. Define t linear forms in wd variables

$$L_i(x_1, \dots, x_{wd}) = \sum_{j=1}^{wd} b_{ij} x_j \in \mathbb{R}[x_1, \dots, x_{wd}] \ \forall\ 1 \leq i \leq t. \tag{5}$$

Our first goal here is to prove the following effective result on density of the image of the set $\Lambda_K(\mathcal{M}) \setminus \mathcal{Z}_{\boldsymbol{S}}$ under the linear forms $L_1, \dots, L_t$ in the torus $\mathbb{R}^t/\mathbb{Z}^t$. Let h denote the usual Weil height on algebraic numbers, as well as its extension to vectors with algebraic coordinates; we recall the definition of height along with other necessary notation in Section 2.

Theorem 1.1. *Let $\boldsymbol{a} = (a_1, \dots, a_t) \in \mathbb{R}^t$ and $\varepsilon > 0$. There exist $\boldsymbol{x} \in \Lambda_K(\mathcal{M}) \setminus \mathcal{Z}_{\boldsymbol{S}}$ and $\boldsymbol{p} \in \mathbb{Z}^t$ such that*

$$|L_i(\boldsymbol{x}) - a_i - p_i| < \varepsilon$$

and

$$|\boldsymbol{x}| \leq \mathfrak{a}_K(t,\ell,s) \left(sdM_{\boldsymbol{S}}|\mathcal{D}_K(\mathcal{M})|^{\frac{s}{2}}\right)^{\mathfrak{K}+1} \left((wd)^{\frac{3}{2}} h(B)\right)^{\mathfrak{K}} \mathfrak{c}_K(\mathcal{M},\ell,t)\ \varepsilon^{-\ell+1},$$

where the exponent $\mathfrak{K} = \ell^2(t+1) - \ell$ and the constants are

$$\mathfrak{a}_K(t,\ell,s) = 2^{\ell t(\ell-1)+sr_1\mathfrak{K}+\frac{sd-1}{2}}(t+1)^{3\ell-1}(t!)^{2\ell}$$

and

$$\mathfrak{c}_K(\mathcal{M},\ell,t) = \min\left\{h(\alpha)^{(\mathfrak{K}+1)sd-1} h(\alpha^{-1})^{\mathfrak{K}}\ :\ \alpha \in \mathfrak{U}_K(\mathcal{M})\right\}.$$

One special case of Theorem 1.1 is when $\mathcal{Z}_{\boldsymbol{S}}$ is a union of linear spaces, which means that the point $\boldsymbol{x}$ in question is in $\Lambda_K(\mathcal{M})$ but outside of a union of sublattices of smaller rank than $\Lambda_K(\mathcal{M})$. What if the rank of such sublattices is equal to the rank of $\Lambda_K(\mathcal{M})$? The next theorem addresses this situation.

Theorem 1.2. *Let $\boldsymbol{a} = (a_1, \dots, a_t) \in \mathbb{R}^t$ and $\varepsilon > 0$. Let $m > 0$ and $\Gamma_1, \dots, \Gamma_m \subset \Lambda_K(\mathcal{M})$ be proper sublattices of full rank and respective determinants $\mathcal{D}_1, \dots, \mathcal{D}_m$, and let $\mathcal{D} = \mathcal{D}_1 \cdots \mathcal{D}_m$. Then for every $\alpha \in \mathfrak{U}_K(\mathcal{M})$ there exist $\boldsymbol{x} \in \Lambda_K(\mathcal{M}) \setminus \bigcup_{i=1}^{m} \Gamma_i$ and $\boldsymbol{p} \in \mathbb{Z}^t$ such that*

$$|L_i(\boldsymbol{x}) - a_i - p_i| < \varepsilon$$

and

$$|\boldsymbol{x}| \leq \left(\mathfrak{b}_K(t,\ell,s,w) \left(h(\alpha)h(\alpha^{-1})h(B)\mathcal{E}_\alpha \right)^{\mathfrak{K}} \frac{\mathcal{D}\ \varepsilon^{-\ell+1}}{|\mathcal{D}_K(\mathcal{M})|^{\frac{sm}{2}}} + 1 \right) \mathcal{E}_\alpha,$$

where the exponent $\mathfrak{K} = \ell^2(t+1) - \ell$*, as in Theorem 1.1, the constant*

$$\mathfrak{b}_K(t,\ell,s,w) = 2^{\ell t(\ell-1)+\frac{\mathfrak{K}}{2}+smr_2}(t+1)^{3\ell-1}(t!)^{2\ell}(wd)^{\frac{3\mathfrak{K}}{2}},$$

and $\mathcal{E}_\alpha =$

$$\mathcal{E}_\alpha(\mathcal{M},\Gamma_1,\dots,\Gamma_m) := 2^{\frac{sr_1-1}{2}} h(\alpha)^{sd-1} |\mathcal{D}_K(\mathcal{M})|^{\frac{s}{2}} \left(\sum_{i=1}^m \frac{\mathcal{D}}{\mathcal{D}_i} - m + 1 \right) + \mathcal{D}^{\frac{1}{sd}}. \tag{6}$$

Here is a sketch of the proofs of Theorems 1.1 and 1.2. We first construct a point $\boldsymbol{y} \in \Lambda_K(\mathcal{M})$ of controlled sup-norm, which is outside of $\mathcal{Z}_{\boldsymbol{S}}$ or $\bigcup_{i=1}^m \Gamma_i$, respectively: in the first case, we use the classical Minkowski's Successive Minima Theorem and a version of Alon's Combinatorial Nullstellensatz [1] (we use the convenient formulation developed in [13]), while in the second we employ a recent result of Henk and Thiel [21] on points of small norm in a lattice outside of a union of full-rank sublattices. We use $\boldsymbol{y}$ to construct an infinite sequence of points $n\boldsymbol{y}$ satisfying the above conditions, and use an effective version of Kronecker's original theorem to obtain a value of the index n (depending on $\varepsilon > 0$) for which the required inequalities on values of linear forms are satisfied. In other words, our avoidance strategy is to follow the line $n\boldsymbol{y}$ until a necessary point is found. One may wish to use a similar strategy, but following a higher dimensional subspace of the ambient space in the hope of a better bound, however it is difficult to guarantee avoiding our fixed algebraic set with such strategy. A convenient effective version of Kronecker's theorem that we use is worked out in Section 3. It should be remarked that the most important feature of approximation results such as our Theorems 1.1 and 1.2 is the exponent on ε in the bounds for $|\boldsymbol{x}|$. As we show, this exponent is the same as in the corresponding bound of the effective version of Kronecker's theorem that we use.

In Section 2 we introduce the necessary notation and provide all the details of our setup. We derive an effective version of Kronecker's theorem in Section 3. We then prove Theorem 1.1 in Section 4 and Theorem 1.2 in Section 5. Finally, in Appendix A we discuss the use of algebraic numbers in a construction of badly approximable matrices, which is closely related to Kronecker-type results via standard transference principles.

2. Notation and setup

Let the notation be as in Section 1. Here we introduce some additional notation needed for our algebraic setup. Let the number field K have discriminant $\mathcal{D}_K$, r_1 real embeddings $\sigma_1,\dots,\sigma_{r_1}$ of K, and r_2 conjugate pairs of complex embeddings $\tau_1,\overline{\tau}_1,\dots,\tau_{r_2},\overline{\tau}_{r_2}$, then $d = r_1 + 2r_2$. For each τ_k, write $\Re(\tau_k)$ for its real part and $\Im(\tau_k)$ for its imaginary part. Let us write $M(K)$ for the set of all places of K, then the archimedean places of K are in correspondence with the embeddings of K, and we choose the absolute values $|\ |_{v_1},\dots,|\ |_{v_{r_1+r_2}}$ so that for each $a \in K$

$$|a|_{v_k} = |\sigma_k(a)|\ \forall\ 1 \leq k \leq r_1$$

and

$$|a|_{v_{r_1+k}} = |\tau_k(a)| = \sqrt{\Re(\tau_k(a))^2 + \Im(\tau_k(a))^2}\ \forall\ 1 \leq k \leq r_2,$$

where $|\ |$ stands for the usual absolute value on $\mathbb{R}$ or $\mathbb{C}$, respectively. For each $v \in M(K)$, we write K_v for the completion of K at v, and for each $n \geq 1$ we define a local norm $|\ |_v : K_v^n \to \mathbb{R}$ by

$$|\boldsymbol{a}|_v := \max_{1 \leq j \leq n} |a_j|_v,$$

for each $\boldsymbol{a} = (a_1, \dots, a_n) \in K_v^n$. Then the extended Weil height on K^n is given by

$$h(\boldsymbol{a}) = \prod_{v \in M(K)} \max\{1, |\boldsymbol{a}|_v\}^{d_v/d},$$

where $d_v = [K_v : \mathbb{Q}_v]$ is the local degree of K at v, so that $\sum_{v|u} d_v = d$ for each $u \in M(\mathbb{Q})$.

For each integer $n \geq 1$, define the standard Minkowski embedding $\rho_K^n : K^n \to \mathbb{R}^{nd}$ by

$$\rho_K^n(\boldsymbol{a}) := \left(\sigma_1^n(\boldsymbol{a}), \dots, \sigma_{r_1}^n(\boldsymbol{a}), \Re(\tau_1^n(\boldsymbol{a})), \Im(\tau_1^n(\boldsymbol{a})), \dots, \Re(\tau_{r_2}^n(\boldsymbol{a})), \Im(\tau_{r_2}^n(\boldsymbol{a}))\right).$$

We will now use Minkowski embedding to construct lattices from $\mathcal{O}_K$-modules and outline some of their main properties; see [14] for further details. Let $1 \leq s \leq w$ be integers, and let $\mathcal{M} \subset K^w$ be an $\mathcal{O}_K$-module such that $\mathcal{M} \otimes_K K \cong K^s$. By the structure theorem for finitely generated projective modules over Dedekind domains (see, for instance [25]),

$$\mathcal{M} = \left\{ \sum_{j=1}^{s} \beta_j \boldsymbol{y}_j : \boldsymbol{y}_j \in \mathcal{O}_K^w,\ \beta_j \in \mathcal{I}_j \right\}$$

for some $\mathcal{O}_K$-fractional ideals $\mathcal{I}_1, \dots, \mathcal{I}_s$ in K. By Proposition 13 on p.66 of [25], the discriminant of $\mathcal{M}$ is then

$$\mathcal{D}_K(\mathcal{M}) := \mathcal{D}_K \prod_{j=1}^{s} \mathbb{N}(\mathcal{I}_j)^2, \tag{7}$$

where $\mathbb{N}(\mathcal{I}_j)$ is the norm of the fractional ideal $\mathcal{I}_j$.

Let $\Lambda_K(\mathcal{M}) := \rho_K^w(\mathcal{M})$ be an algebraic lattice of rank sd in $\mathbb{R}^{wd}$, then a direct adaptation of Lemma 2 on p.115 of [25] implies that the determinant of $\Lambda_K(\mathcal{M})$ is

$$\det(\Lambda_K(\mathcal{M})) = 2^{-sr_2} |\mathcal{D}_K(\mathcal{M})|^{\frac{s}{2}} = 2^{-sr_2} |\mathcal{D}_K|^{\frac{s}{2}} \prod_{j=1}^{s} \mathbb{N}(\mathcal{I}_j), \tag{8}$$

where the last identity follows by (7) above. Let $\boldsymbol{x} \in \Lambda_K(\mathcal{M})$, then $\boldsymbol{x} = \rho_K^w(\boldsymbol{a})$ for some $\boldsymbol{a} \in \mathcal{M}$ and

$$|\boldsymbol{x}| \geq \frac{1}{\sqrt{2}} h(\alpha)^{-1}, \tag{9}$$

for any $\alpha \in \mathfrak{U}_K(\mathcal{M})$ by inequality (54) of [14]. Let $v \in M(K)$ be an archimedean place, and assume first that it corresponds to a real embedding σ_j for some $1 \leq j \leq r_1$, then $|\boldsymbol{a}|_v = |\boldsymbol{x}|$. On the other hand, if v corresponds to a complex embedding τ_j for some $1 \leq j \leq r_2$, then $|\boldsymbol{a}|_v \leq \left(\sum_{j=1}^{wd} x_j^2\right)^{1/2} \leq \sqrt{wd}\ |\boldsymbol{x}|$. Hence for each $v \mid \infty$,

$$|\boldsymbol{x}| \leq |\boldsymbol{a}|_v \leq \sqrt{wd}\ |\boldsymbol{x}|. \tag{10}$$

Let $L_1, \dots, L_t$ be the linear forms defined in (5). For each $1 \leq i \leq t$, we define

$$|L_i|_v = \max_{1 \leq j \leq wd} |b_{ij}|_v,$$

for each place $v \in M(E)$, and define the height of L_i to be

$$h(L_i) = H(1, b_{i1}, \dots, b_{i(wd)}) = \prod_{v \in M(E)} \max\{1, |L_i|_v\}^{\ell_v/\ell}.$$

We similarly define the height of the matrix B to be

$$h(B) = H(1, b_{11}, \dots, b_{t(wd)}),$$

then $h(L_i) \leq h(B)$ for all $1 \leq i \leq t$. We are now ready to proceed.

3. An effective version of Kronecker's theorem

In this section we derive an effective version of Kronecker's theorem, which we then use to prove Theorems 1.1 and 1.2. Similar to the setup in the beginning of Section 1, let $1, \theta_1, \dots, \theta_t$ be $\mathbb{Q}$-linearly independent real algebraic numbers. For each $1 \leq j \leq t$, let $f_j(x) \in \mathbb{Z}[x]$ be the minimal polynomial of θ_j of degree d_j, $|f_j|$ be the maximum of absolute values of the coefficients of f_j, and A_j be the leading coefficient of f_j, so $A_j \leq |f_j|$. By Lemma 3.11 of [32],

$$\frac{1}{2^{d_j}}\ |f_j| \leq h(\theta_j)^{d_j} \leq \sqrt{d_j+1}\ |f_j|,$$

for every $1 \leq j \leq t$. Define A to be the least common multiple of $A_1, \dots, A_t$, so

$$A \leq \prod_{j=1}^{t} |f_j| \leq \prod_{j=1}^{t} (2h(\theta_j))^{d_j}. \tag{11}$$

Let $F = \mathbb{Q}(\theta_1, \dots, \theta_t)$ be a number field of degree $e \geq t+1$, then $e \leq \prod_{j=1}^{t} d_j$. Let $\theta_{t+1}, \dots, \theta_{e-1} \in F$ be such that

$$1 = \theta_0, \theta_1, \dots, \theta_t, \theta_{t+1}, \dots, \theta_{e-1}$$

form a $\mathbb{Q}$-basis for F. Let $\sigma_1, \dots, \sigma_e$ be the embeddings of F into $\mathbb{C}$. We recall Liouville inequality. For any $\boldsymbol{m} = (m_0, \dots, m_t, 0, \dots, 0) \in \mathbb{Z}^e$,

$$A^e \prod_{i=1}^{e} \left| \sum_{j=0}^{e-1} \sigma_i(\theta_j) m_j \right| \geq 1, \tag{12}$$

and so

$$A^e \left((t+1) \max_{1 \leq i \leq e, 0 \leq j \leq t} |\sigma_i(\theta_j)| \right)^{e-1} |\boldsymbol{m}|^{e-1} \| m_1\theta_1 + \dots + m_t\theta_t \| \geq 1. \tag{13}$$

Now observe that

$$\max_{1 \leq i \leq e, 0 \leq j \leq t} |\sigma_i(\theta_j)| \leq \max_{1 \leq j \leq t} h(\theta_j)^{d_j},$$

and so define

$$\mathcal{C}_1 = \mathcal{C}_1(\theta_1, \dots, \theta_t) := \left((t+1) \max_{1 \leq j \leq t} h(\theta_j)^{d_j} \right)^{e-1} \prod_{j=1}^{t} (2h(\theta_j))^{ed_j}. \tag{14}$$

Then for any $\mathbf{0} \neq \boldsymbol{m} \in \mathbb{Z}^t$,

$$\| m_1\theta_1 + \dots + m_t\theta_t \| \geq \mathcal{C}_1^{-1} |\boldsymbol{m}|^{-e+1}. \tag{15}$$

We will now apply a transference homogeneous-inhomogeneous argument. A transference principle of this sort was first described in Chapter V, §4 of [3]; the particular stronger result we are applying here is obtained in [2]. Let us write

$$M(\boldsymbol{y}) = \sum_{i=1}^{t} \theta_i y_i$$

for $\boldsymbol{y} = (y_1, \dots, y_t) \in \mathbb{Z}^t$, and let

$$L_j(x) = \theta_j x, \ 1 \le j \le t$$

for $x \in \mathbb{Z}$. Then (15) guarantees that for any $\boldsymbol{0} \neq \boldsymbol{y} \in \mathbb{Z}^t$ with $|\boldsymbol{y}| \le Y$,

$$\|M(\boldsymbol{y})\| \ge \mathcal{C}_1^{-1} Y^{-(e-1)}.$$

Now applying the transference Lemma 3 of [2] to these linear forms, we have that for every $\boldsymbol{a} = (a_1, \dots, a_t) \in \mathbb{R}^t$ there exists $x \in \mathbb{Z}$ such that $|x| \le 2^{-t}((t+1)!)^2 \mathcal{C}_1 Y^{e-1}$ and

$$\max_{1 \le j \le t} \|L_j(x) - a_j\| \le 2^{-t}((t+1)!)^2 Y^{-1}.$$

Letting $Q = \left(2^t((t+1)!)^{-2} Y\right)^{e-1}$, we obtain that

$$\max_{1 \le j \le t} \|L_j(x) - \alpha_j\| \le Q^{-\frac{1}{e-1}}$$

for some $0 \neq x \in \mathbb{Z}$ with $|x| \le 2^{-et}((t+1)!)^{2e} \mathcal{C}_1 Q$. Taking $\varepsilon = Q^{-\frac{1}{e-1}}$ immediately yields the following effective version of Kronecker's theorem.

Theorem 3.1. *Let $1, \theta_1, \dots, \theta_t$ be $\mathbb{Q}$-linearly independent real algebraic numbers, and let $e = [\mathbb{Q}(\theta_1, \dots, \theta_t) : \mathbb{Q}]$. Let $\mathcal{C}_1$ be given by (14) above, and let $\varepsilon > 0$. Then for any $(a_1, \dots, a_t) \in \mathbb{R}^t$ there exists $q \in \mathbb{Z} \setminus \{0\}$ such that*

$$\|q\theta_j - a_j\| \le \varepsilon, \ 1 \le j \le t \tag{16}$$

and

$$|q| \le 2^{-et}((t+1)!)^{2e} \mathcal{C}_1 \varepsilon^{-e+1}.$$

In particular, if $h(\theta_j) \le H$ for all $1 \le j \le t$ and $\max\{e, d_1, \dots, d_t\} \le \ell$, then

$$|q| \le \left(2^{\ell t(\ell-1)}(t+1)^{3\ell-1}(t!)^{2\ell} H^{\ell^2(t+1)-\ell}\right) \varepsilon^{-\ell+1}.$$

Remark 3.1. Stronger non-effective results can be derived as corollaries of Schmidt's Subspace Theorem. For instance, results discussed in Chapter 6, §2 of [30] together with the transference principles of Chapter V, §4 of [3] and [2] imply, for any $\varepsilon > 0$ and $\boldsymbol{a} \in \mathbb{R}^t$ under the assumptions of Theorem 3.1, the existence of $q \in \mathbb{Z}$ satisfying 16 such that

$$|q| \le \mathcal{C}'(\delta) \varepsilon^{-t-\delta},$$

for any $\delta > 0$, where the constant $\mathcal{C}'(\delta)$ is non-effective. This would result in the same exponent on ε in the bounds for $|q|$ in Theorems 1.1 and 1.2, but with non-effective constants.

4. Proof of Theorem 1.1

Here we present the proof of our first result. Since $\Lambda_K(\mathcal{M}) \nsubseteq \mathcal{Z}_{\boldsymbol{S}}$, $\Lambda_K(\mathcal{M}) \nsubseteq Z(\mathcal{S}_i)$ for all $1 \leq i \leq m$, and so for each i at least one polynomial P_i in $\mathcal{S}_i$ is not identically zero on $\Lambda_K(\mathcal{M})$. Clearly for each $1 \leq i \leq m$,

$$Z(\mathcal{S}_i) \subseteq Z(P_i) := \left\{ \boldsymbol{x} \in \mathbb{R}^{wd} : P_i(\boldsymbol{x}) = 0 \right\}.$$

Define

$$P(\boldsymbol{x}) = \prod_{i=1}^{m} P_i(\boldsymbol{x}),$$

so that $\Lambda_K(\mathcal{M}) \nsubseteq Z(P)$ while $\mathcal{Z}_{\boldsymbol{S}} \subseteq Z(P)$ and $\deg(P) \leq M_{\boldsymbol{S}}$. We will next construct a point $\boldsymbol{y} \in \Lambda_K(\mathcal{M})$ of controlled sup-norm such that $P(\boldsymbol{y}) \neq 0$.

Let $V = \operatorname{span}_{\mathbb{R}} \Lambda_K(\mathcal{M})$ be the sd-dimensional subspace of $\mathbb{R}^{wd}$ spanned by the lattice $\Lambda_K(\mathcal{M})$. For a positive real number μ, let us write

$$C_V(\mu) := \{\boldsymbol{x} \in V : |\boldsymbol{x}| \leq \mu\}$$

for the sd-dimensional cube with side-length 2μ centered at the origin in V, so $C_V(\mu) = \mu C_V(1)$. Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{sd}$ be the successive minima of $\Lambda_K(\mathcal{M})$ with respect to the cube $C_V(1)$. In other words, for each $1 \leq i \leq sd$,

$$\lambda_i := \min \left\{ \mu \in \mathbb{R}_{>0} : \dim_{\mathbb{R}} \operatorname{span}_{\mathbb{R}} \left(\Lambda_K(\mathcal{M}) \cap C_V(\mu) \right) \geq i \right\}.$$

Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{sd}$ be a collection of linearly independent vectors in $\Lambda_K(\mathcal{M})$ corresponding to these successive minima, then $|\boldsymbol{v}_i| = \lambda_i$. Since the volume of sd-dimensional cube $C_V(1)$ is 2^{sd}, Minkowski's Successive Minima Theorem (see, for instance, [4] or [19]) implies that

$$\frac{\det(\Lambda_K(\mathcal{M}))}{(sd)!} \leq \prod_{i=1}^{sd} |\boldsymbol{v}_i| \leq \det(\Lambda_K(\mathcal{M})),$$

where $\frac{1}{\sqrt{2}} h(\alpha)^{-1} \leq |\boldsymbol{v}_1| \leq \cdots \leq |\boldsymbol{v}_{sd}|$, by (9). This means that

$$|\boldsymbol{v}_1| \leq \cdots \leq |\boldsymbol{v}_{sd}| \leq \left(\sqrt{2} h(\alpha) \right)^{sd-1} \det(\Lambda_K(\mathcal{M})). \tag{17}$$

Let $I(M_{\boldsymbol{S}}) = \{0, 1, 2, \dots, M_{\boldsymbol{S}}\}$ be the set of the first $M_{\boldsymbol{S}} + 1$ non-negative integers. For each $\boldsymbol{\xi} \in I(M_{\boldsymbol{S}})^{sd}$, define

$$\boldsymbol{v}(\boldsymbol{\xi}) = \sum_{i=1}^{sd} \xi_i \boldsymbol{v}_i,$$

then

$$|\boldsymbol{v}(\boldsymbol{\xi})| = \max_{1 \leq j \leq wd} \left| \sum_{i=1}^{sd} \xi_i v_{ij} \right| \leq sd |\boldsymbol{\xi}| |\boldsymbol{v}_{sd}| \leq sd M_{\boldsymbol{S}} \left(\sqrt{2} h(\alpha) \right)^{sd-1} \det(\Lambda_K(\mathcal{M})), \tag{18}$$

by (17). Assume that $P(\boldsymbol{v}(\boldsymbol{\xi})) = 0$ for each $\boldsymbol{\xi} \in I(M_{\boldsymbol{S}})^{sd}$. Then Theorem 4.2 of [13] implies that $P(\boldsymbol{x})$ must be identically zero on V, which would contradict the fact that P does not vanish identically on $\Lambda_K(\mathcal{M})$. Hence there must exist some $\boldsymbol{\xi} \in I(M_{\boldsymbol{S}})^{sd}$ such that P does not vanish at the corresponding $\boldsymbol{y} := \boldsymbol{v}(\boldsymbol{\xi})$, and $|\boldsymbol{y}| \leq sd M_{\boldsymbol{S}} \left(\sqrt{2} h(\alpha) \right)^{sd-1} \det(\Lambda_K(\mathcal{M}))$ by (18). Since $P(\boldsymbol{x})$ is a homogeneous polynomial, it must be true that $P(n\boldsymbol{y}) \neq 0$ for every $n \in \mathbb{Z}_{>0}$. On the other hand, by our construction

$$n\boldsymbol{y} = n \sum_{i=1}^{sd} \xi_i \boldsymbol{v}_i \in \operatorname{span}_{\mathbb{Z}} \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_{sd}\} \subseteq \Lambda_K(\mathcal{M}),$$

and so $\{n\boldsymbol{y}\}_{n\in\mathbb{Z}_{>0}}$ gives an infinite sequence of points in $\Lambda_K(\mathcal{M})$ outside of $\mathcal{Z}_{\boldsymbol{S}}$. For each such point, we have

$$L_i(n\boldsymbol{y}) = nL_i(\boldsymbol{y}),\ \forall\ 1 \leq i \leq t.$$

Let us define, for each $1 \leq i \leq t$,

$$\theta_i := L_i(\boldsymbol{y}) = \sum_{j=1}^{wd} b_{ij} y_j \neq 0, \tag{19}$$

since $y_j \in K_1$, not all zero, and b_{ij} are K_1-linearly independent. Notice that $\theta_1, \dots, \theta_t \in E$, and hence all of them are algebraic numbers of degree $\leq \ell$.

Let $\alpha \in \mathfrak{U}_K(\mathcal{M})$. Then, by (10), for each archimedean $v \in M(E)$,

$$\begin{aligned} \max\{1, |\theta_i|_v\} &\leq \max\{1, (wd)^{\frac{3}{2}} |L_i|_v |\boldsymbol{y}|\} \leq (wd)^{\frac{3}{2}} \max\{1, |\boldsymbol{y}|\} \max\{1, |L_i|_v\} \\ &\leq \sqrt{2}\ (wd)^{\frac{3}{2}} h(\alpha) |\boldsymbol{y}| \max\{1, |L_i|_v\}, \end{aligned} \tag{20}$$

by (9). By (18), $|\boldsymbol{y}| \leq sdM_{\boldsymbol{S}} \left(\sqrt{2}h(\alpha)\right)^{sd-1} \det(\Lambda_K(\mathcal{M}))$, and hence

$$\max\{1, |\theta_i|_v\} \leq sd(wd)^{\frac{3}{2}} M_{\boldsymbol{S}} \left(\sqrt{2}h(\alpha)\right)^{sd} \det(\Lambda_K(\mathcal{M})) \max\{1, |L_i|_v\}. \tag{21}$$

Now suppose $v \in M(E)$ is non-archimedean. Then αy_j is an algebraic integer for each $1 \leq j \leq wd$, and hence $|\alpha y_j|_v = |\alpha|_v |y_j|_v \leq 1$, meaning that

$$\max\{1, |y_1|_v, \dots, |y_{wd}|_v\} \leq \max\{1, |\alpha|_v^{-1}\}.$$

Then

$$\begin{aligned} \max\{1, |\theta_i|_v\} &\leq \max\{1, |L_i|_v\} \max\{1, |y_1|_v, \dots, |y_{wd}|_v\} \\ &\leq \max\{1, |\alpha^{-1}|_v\} \max\{1, |L_i|_v\}, \end{aligned} \tag{22}$$

for each non-archimedean $v \in M(E)$. Taking a product over all places of E, we obtain:

$$\begin{aligned} h(\theta_i) &= \prod_{v\in M(E)} \max\{1, |\theta_i|_v\}^{\frac{\ell_v}{\ell}} = \left(\prod_{v|\infty} \max\{1, |\theta_i|_v\}^{\ell_v} \times \prod_{v\nmid\infty} \max\{1, |\theta_i|_v\}^{\ell_v} \right)^{\frac{1}{\ell}} \\ &\leq sd(wd)^{\frac{3}{2}} M_{\boldsymbol{S}} \left(\sqrt{2}h(\alpha)\right)^{sd} \det(\Lambda_K(\mathcal{M})) h(L_i) \prod_{v\nmid\infty} \max\{1, |\alpha^{-1}|_v\}^{\frac{\ell_v}{\ell}} \\ &\leq sd(wd)^{\frac{3}{2}} M_{\boldsymbol{S}} \left(\sqrt{2}h(\alpha)\right)^{sd} h(\alpha^{-1}) \det(\Lambda_K(\mathcal{M})) h(L_i). \end{aligned}$$

Recalling that $h(L_i) \leq h(B)$ for all $1 \leq i \leq t$, we obtain

$$h(\theta_i) \leq 2^{\frac{sd}{2}} sd(wd)^{\frac{3}{2}} M_{\boldsymbol{S}} h(\alpha)^{sd} h(\alpha^{-1}) \det(\Lambda_K(\mathcal{M})) h(B), \tag{23}$$

for each $1 \leq i \leq t$, where the choice of $\alpha \in \mathfrak{U}_K(\mathcal{M})$ is arbitrary.

We will now show that $1, \theta_1, \dots, \theta_t$ are $\mathbb{Q}$-linearly independent. Suppose not, then there exist $c_0, c_1, \dots, c_t \in \mathbb{Q}$, not all zero, such that

$$c_0 = \sum_{i=1}^{t} c_i \theta_i = \sum_{i=1}^{t} \sum_{j=1}^{wd} c_i y_j b_{ij},$$

where not all $c_i y_j$ are equal to zero. Recall that $\boldsymbol{y} \in \Lambda_K(\mathcal{M})$, meaning that coordinates of $\boldsymbol{y}$ are in K_1, hence all $c_i y_j$ are in K_1. This contradicts the assumption that $1, b_{11}, \dots, b_{1(wd)}$ are linearly independent over K_1. Hence $1, \theta_1, \dots, \theta_t$ must be linearly independent over $\mathbb{Q}$.

Now let $\boldsymbol{a} = (a_1, \dots, a_t) \in \mathbb{R}^t$ and $\varepsilon > 0$, as in the statement of our theorem. Then, by (23) and Theorem 3.1, there exists $q \in \mathbb{Z}$ and $\boldsymbol{p} \in \mathbb{Z}^t$ such that

$$\begin{aligned} |q| \quad &\leq \quad 2^{\ell t(\ell-1)} (t+1)^{3\ell-1} (t!)^{2\ell} \times \\ &\times \left(2^{\frac{sd}{2}} sd (wd)^{\frac{3}{2}} M_{\boldsymbol{S}} h(\alpha)^{sd} h(\alpha^{-1}) \det(\Lambda_K(\mathcal{M})) h(B) \right)^{\ell^2(t+1)-\ell} \varepsilon^{-\ell+1} \end{aligned} \tag{24}$$

and

$$|q\theta_i - a_i - p_i| < \varepsilon \ \forall \ 1 \leq i \leq t.$$

Letting $\boldsymbol{x} = q\boldsymbol{y}$, we see that $q\theta_i = L_i(\boldsymbol{x})$ for each $1 \leq i \leq t$ and $|\boldsymbol{x}| = |q||\boldsymbol{y}|$. Combining these observations with (18), (24) and (8) and taking a minimum over all $\alpha \in \mathfrak{U}_K(\mathcal{M})$ finishes the proof of the theorem.

5. Proof of Theorem 1.2

Let $\Gamma_1, \dots, \Gamma_m$ be full-rank sublattices of $\Lambda_K(\mathcal{M})$ of respective determinants $\mathcal{D}_1, \dots, \mathcal{D}_m$. Let $\Omega = \cap_{i=1}^m \Gamma_i$, then Ω also has full rank and

$$\mathcal{D} := \mathcal{D}_1 \cdots \mathcal{D}_m \geq \det \Omega.$$

We write λ_i for the successive minima of $\Lambda_K(\mathcal{M})$ and $\lambda_i(\Omega)$ for the successive minima of Ω. Theorem 1.2 of [21] implies that there exists $\boldsymbol{y} \in \Lambda_K(\mathcal{M}) \setminus \bigcup_{i=1}^m \Gamma_i$ such that

$$|\boldsymbol{y}| < \frac{\det \Lambda_K(\mathcal{M})}{\lambda_1(\Omega)^{sd-1}} \left(\sum_{i=1}^{m} \frac{\mathcal{D}}{\mathcal{D}_i} - m + 1 \right) + \lambda_1(\Omega).$$

Our first goal is to make this bound more explicit in terms of the parameters of $\mathcal{M}$. First notice that by Minkowski's Successive Minima Theorem,

$$\lambda_1(\Omega) \leq \left(\prod_{i=1}^{sd} \lambda_i(\Omega) \right)^{1/sd} \leq (\det \Omega)^{1/sd} \leq \mathcal{D}^{1/sd}.$$

We also need a lower bound on $\lambda_1(\Omega)$. Observe that $\lambda_1(\Omega) \geq \lambda_1$, while $\lambda_1 \geq \frac{1}{\sqrt{2}} h(\alpha)^{-1}$ for any $\alpha \in \mathfrak{U}_K(\mathcal{M})$, by (9) above. Putting these estimates together, we see that

$$|\boldsymbol{y}| < \left(\sqrt{2} h(\alpha) \right)^{sd-1} \det \Lambda_K(\mathcal{M}) \left(\sum_{i=1}^{m} \frac{\mathcal{D}}{\mathcal{D}_i} - m + 1 \right) + \mathcal{D}^{1/sd} \tag{25}$$

for any $\alpha \in \mathfrak{U}_K(\mathcal{M})$.

Since $\boldsymbol{y} \in \Lambda_K(\mathcal{M})$ and $|\Lambda_K(\mathcal{M}) : \Gamma_i| = \mathcal{D}_i / \det \Lambda_K(\mathcal{M})$ for each $1 \leq i \leq m$, it follows that

$$(g|\Lambda_K(\mathcal{M}) : \Gamma_i|)\, \boldsymbol{y} = \frac{g\mathcal{D}_i}{\det \Lambda_K(\mathcal{M})} \boldsymbol{y} \in \Gamma_i,$$

for every $g \in \mathbb{Z}$, and hence

$$\left(\frac{g\mathcal{D}_1 \cdots \mathcal{D}_m}{(\det \Lambda_K(\mathcal{M}))^m} \right) \boldsymbol{y} = \left(\frac{g\mathcal{D}}{(\det \Lambda_K(\mathcal{M}))^m} \right) \boldsymbol{y} \in \Omega,$$

for every $g \in \mathbb{Z}$. Therefore, it must be true that

$$\left(\frac{g\mathcal{D}}{(\det \Lambda_K(\mathcal{M}))^m} + 1 \right) \boldsymbol{y} \in \Lambda_K(\mathcal{M}) \setminus \bigcup_{i=1}^m \Gamma_i,$$

for every $g \in \mathbb{Z}$. For brevity, let us write $\mathcal{D}' = \frac{\mathcal{D}}{(\det \Lambda_K(\mathcal{M}))^m}$.

From here on, the argument is largely similar to the proof of Theorem 1.1 above, but with some notable changes. For each $1 \leq i \leq t$, let θ_i be as in (19) for our choice of $\boldsymbol{y} \in \Lambda_K(\mathcal{M}) \setminus \bigcup_{i=1}^m \Gamma_i$ satisfying (25) as above, then

$$L_i((g\mathcal{D}' + 1)\boldsymbol{y}) = (g\mathcal{D}' + 1)\theta_i \ \forall \ 1 \leq i \leq t.$$

Using (20) with (25) instead of (18), we obtain that $\max\{1, |\theta_i|_v\} \leq$

$$(wd)^{\frac{3}{2}} \left(\left(\sqrt{2} h(\alpha) \right)^{sd} \det \Lambda_K(\mathcal{M}) \left(\sum_{i=1}^m \frac{\mathcal{D}}{\mathcal{D}_i} - m + 1 \right) + \mathcal{D}^{\frac{1}{sd}} \sqrt{2} h(\alpha) \right) \max\{1, |L_i|_v\}$$

for all archimedean $v \in M(E)$, while for the non-archimedean $v \in M(E)$,

$$\max\{1, |\theta_i|_v\} \leq \max\{1, |\alpha^{-1}|_v\} \max\{1, |L_i|_v\},$$

as in (22). Taking the product over all places of E, we have for every $1 \leq i \leq t$:

$$\begin{aligned} h(\theta_i) \quad &\leq \quad (wd)^{\frac{3}{2}} \sqrt{2} h(\alpha) h(\alpha^{-1}) h(B) \times \\ &\quad \times \left(\left(\sqrt{2} h(\alpha) \right)^{sd-1} \det \Lambda_K(\mathcal{M}) \left(\sum_{i=1}^m \frac{\mathcal{D}}{\mathcal{D}_i} - m + 1 \right) + \mathcal{D}^{\frac{1}{sd}} \right), \end{aligned} \tag{26}$$

and $1, \theta_1, \dots, \theta_t$ (and hence $1, \mathcal{D}'\theta_1, \dots, \mathcal{D}'\theta_t$) are $\mathbb{Q}$-linearly independent by the same reasoning as in the proof of Theorem 1.1.

Now let $\boldsymbol{a} = (a_1, \dots, a_t) \in \mathbb{R}^t$ and $\varepsilon > 0$, as in the statement of our theorem. Notice that for each $1 \leq i \leq t$,

$$|(g\mathcal{D}' + 1)\theta_i - a_i - p_i| = |g(\mathcal{D}'\theta_i) + (\theta_i - a_i) - p_i|,$$

for any integers $p_1, \dots, p_t$. Then, applying Theorem 3.1 to approximate the vector $(\theta_1 - a_1, \dots, \theta_t - a_t)$ by the fractional parts of the integer multiples of the vector $(\mathcal{D}'\theta_1, \dots, \mathcal{D}'\theta_t)$, we conclude that there exists $g \in \mathbb{Z}$ and $\boldsymbol{p} \in \mathbb{Z}^t$ such that

$$\begin{aligned} |g| \quad &\leq \quad 2^{\ell t(\ell-1)} (t+1)^{3\ell-1} (t!)^{2\ell} \times \\ &\quad \times \left((wd)^{\frac{3}{2}} \sqrt{2} h(\alpha) h(\alpha^{-1}) h(B) \mathcal{E}_\alpha(\mathcal{M}, \Gamma_1, \dots, \Gamma_m) \right)^{\ell^2(t+1)-\ell} \varepsilon^{-\ell+1}, \end{aligned} \tag{27}$$

where $\mathcal{E}_\alpha(\mathcal{M}, \Gamma_1, \dots, \Gamma_m)$ is as in (6), and

$$|g(\mathcal{D}'\theta_i) + (\theta_i - a_i) - p_i| < \varepsilon \ \forall \ 1 \leq i \leq t.$$

Letting $\boldsymbol{x} = (g\mathcal{D}' + 1)\boldsymbol{y}$, we see that $(g\mathcal{D}' + 1)\theta_i = L_i(\boldsymbol{x})$ for each $1 \leq i \leq t$ and $|\boldsymbol{x}| = |g\mathcal{D}' + 1||\boldsymbol{y}|$. Combining these observations with (25), (27) and (8) finishes the proof of the theorem.

Appendix A. Badly approximable matrices and related topics

Algebraic numbers give effective constructions of badly approximable matrices. Systems of linear forms corresponding to badly approximable matrices admit the best and optimal results for the Kronecker-type setting.

1. Badly approximable matrices. We consider an $m \times n$ matrix

$$\Theta = \left(\begin{array}{ccc} \theta_{1,1} & \dots & \theta_{1,m} \\ \dots & \dots & \dots \\ \theta_{n,1} & \dots & \theta_{n,m} \end{array} \right)$$

with real entries $\theta_{i,j}$ and the corresponding system of linear forms

$$L_i(\boldsymbol{x}) = \sum_{j=1}^{m} \theta_{i,j} x_j, \ \ 1 \le i \le n, \ \ \boldsymbol{x} = (x_1, ..., x_m).$$

Suppose that for any nonzero integer vector $\boldsymbol{x}$ one has

$$\max_{1 \le i \le n} ||L_i(\boldsymbol{x})|| \ne 0.$$

Then, by Dirichlet theorem there exist infinitely many primitive integer vectors $\boldsymbol{x} \in \mathbb{Z}^m$ such that

$$\max_{1 \le i \le n} ||L_i(\boldsymbol{x})|| \le \left(\max_{1 \le j \le m} |x_j| \right)^{-\frac{m}{n}}.$$

A matrix Θ is defined to be *badly approximable* if there exists a positive constant $\gamma = \gamma(\Theta)$ such that

$$\max_{1 \le i \le n} ||L_i(\boldsymbol{x})|| \ge \gamma \left(\max_{1 \le j \le m} |x_j| \right)^{-\frac{m}{n}}. \tag{28}$$

for all nonzero integer vectors $\boldsymbol{x}$.

The set of badly approximable matrices has Lebesgue measure zero in $\mathbb{R}^{m \times n}$. A classical result by Wolfgang M. Schmidt [29] (see also the book [30]) states that the set of badly approximable matrices is a winning set in $\mathbb{R}^{m \times n}$ and hence it has the full Hausdorff dimension in $\mathbb{R}^{m \times n}$.

Here we discuss an algebraic construction of badly approximeble matrices which should be well-known, however there is an interesting observation related to this construction. In Schmidt's 1969 paper [29] it is written that O. Perron [27] constructed badly approximable $m \times n$ matrices Θ with algebraic elements. If we look carefully at [27], we see that Perron considered the cases $m = 1$ (simultaneous approximation to n numbers, Satz 1 from [27]) and the case $n = 1$ (one linear form in m variables; this case is trivial as the result follows immediately from the lower bound for the norm of an algebraic number, see formulas at the bottom of page 79 from [27]) only. For arbitrary values of m and n we found an example of an algebraic badly approximable matrix in a survey paper by Rauzy [28] without any further references. Here we give a general version of this example.

2. Algebraic matrices. Rauzy's example generalized. Let $\mathbb{K}$ be a totally real number field of degree $d = n + m$, meaning that all of its conjugates fields

$$\mathbb{K} = \mathbb{K}^{(1)}, \mathbb{K}^{(2)}, \dots, \mathbb{K}^{(d)}$$

are real. Let

$$\beta_1, ..., \beta_d \tag{29}$$

be a basis of $\mathbb{K}$ over $\mathbb{Q}$. For each $\xi \in \mathbb{K}$, let

$$\xi = \xi^{(1)} \in \mathbb{K}^{(1)}, \dots, \xi^{(d)} \in \mathbb{K}^{(d)}$$

be its algebraic conjugates. We define the $m \times n$ matrix

$$B_1 = \begin{pmatrix} \beta_1^{(1)} & \dots & \beta_m^{(1)} \\ \dots & \dots & \dots \\ \beta_1^{(n)} & \dots & \beta_m^{(n)} \end{pmatrix}$$

and the $n \times n$ matrix

$$B_2 = \begin{pmatrix} \beta_{m+1}^{(1)} & \dots & \beta_{m+n}^{(1)} \\ \dots & \dots & \dots \\ \beta_{m+1}^{(n)} & \dots & \beta_{m+n}^{(n)} \end{pmatrix}.$$

Proposition 1. *Suppose*[1] *that* $\det B_2 \neq 0$ *and define*

$$\Theta = B_2^{-1} B_1.$$

Then θ is a real $m \times n$ badly approximable matrix.

We give a proof of Proposition 1 in the next subsection. It is a straightforward generalization of the proof from [28]. A standard transference argument (see Chapter V from [3]) shows that the transposed $n \times m$ matrix $\Theta^T = B_1^T (B_2^T)^{-1}$ is badly approximable, however this fact admits a nice algebraic proof which may be of importance. We would like to give it in Subsection 3.

3. Proof of Proposition 1. Let

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}, \ \boldsymbol{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \ \boldsymbol{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}$$

be integer vectors. We consider linear forms

$$\sum_{j=1}^{d} \beta_j^{(\nu)} z_j = \sum_{j=1}^{m} \beta_j^{(\nu)} x_j + \sum_{j=1}^{n} \beta_{m+j}^{(\nu)} y_j, \quad 1 \leq \nu \leq d.$$

From the definitions of matrices B_1 and B_2 we see that

$$\sum_{j=1}^{d} \beta_j^{(\nu)} z_j = \sum_{j=1}^{n} \beta_{m+j}^{(\nu)} (L_j(\boldsymbol{x}) + y_j), \quad 1 \leq \nu \leq n,$$

where L_j denote the linear forms corresponding to the matrix Θ,

Suppose that positive integer A_j is the leading coefficient for the canonical polynomial for β_j and $A = A_1 \cdots A_d$. Then $A\beta_j^{(\nu)}$ is an algebraic integer for any j, ν

[1]given the basis (29) one may order elements in it to satisfy this assertion

and so

$$\begin{aligned} A^{-d} &\leq \prod_{\nu=1}^{d}\left|\sum_{j=1}^{d}\beta_j^{(\nu)}z_j\right| = \prod_{\nu=1}^{n}\left|\sum_{j=1}^{n}\beta_{m+j}^{(\nu)}(L_j(\boldsymbol{x})+y_j)\right| \times \prod_{\nu=n+1}^{d}\left|\sum_{j=1}^{d}\beta_j^{(\nu)}z_j\right| \\ &\leq \varkappa \max_{1\leq i\leq n}||L_i(\boldsymbol{x})||^n \cdot \max_{1\leq j\leq m}|x_j|^m \end{aligned}$$

with

$$\varkappa = \varkappa(\boldsymbol{\beta}) = \prod_{\nu=1}^{n}\sum_{j=1}^{n}|\beta_{m+j}^{(\nu)}| \ \times \prod_{\nu=n+1}^{d}\left(\sum_{j=1}^{m}|\beta_j^{(\nu)}| + \sum_{j=1}^{n}|\beta_{m+j}^{(\nu)}|\left(\sum_{i=1}^{m}|\theta_{i,j}|+\frac{1}{2}\right)\right).$$

Hence we obtain (28) with $\gamma = A^{-d/n}\varkappa^{-1/n}$.

4. Algebraic proof of the dual statement. Now we prove that the matrix Θ^T is an $n\times m$ badly approximable matrix. Consider the dual basis $\omega_1,...,\omega_d\in\mathbb{K}$, so

$$\sum_{\nu=1}^{d}\beta_i^{(\nu)}\omega_j^{(\nu)} = \begin{cases}0, & i\neq j,\\ 1, & i=j.\end{cases} \tag{30}$$

Let

$$B = \begin{pmatrix} \beta_1^{(1)} & ... & \beta_d^{(1)} \\ ... & ... & ... \\ \beta_1^{(d)} & ... & \beta_d^{(d)} \end{pmatrix}, \quad \Omega = \begin{pmatrix} \omega_1^{(1)} & ... & \omega_d^{(1)} \\ ... & ... & ... \\ \omega_1^{(d)} & ... & \omega_d^{(d)} \end{pmatrix}.$$

Equation (30) implies that

$$B^T\Omega = E,$$

or

$$\Omega B^T = E. \tag{31}$$

We define a block $d\times d$ matrix

$$B_0 = \begin{pmatrix} B_1^T & E_{m\times m} \\ B_2^T & \mathbf{0}_{m\times n} \end{pmatrix},$$

where $E_{m\times m}$ is the identity matrix and $\mathbf{0}_{m\times n}$ is the zero matrix. Notice that the first n columns of B^T and B_0 coincide. From (31) it is clear that

$$\Omega_1 = \Omega B_0 = \begin{pmatrix} 1 & ... & 0 & \omega_1^{(1)} & ... & \omega_m^{(1)} \\ ... & ... & ... & ... & ... & ... \\ 0 & ... & 1 & \omega_1^{(n)} & ... & \omega_m^{(n)} \\ 0 & ... & 0 & \omega_1^{(n+1)} & ... & \omega_m^{(n+1)} \\ ... & ... & ... & ... & ... & ... \\ 0 & ... & 0 & \omega_1^{(d)} & ... & \omega_m^{(d)} \end{pmatrix}. \tag{32}$$

For the inverse matrix B_0^{-1} we have

$$B_0^{-1} = \begin{pmatrix} \mathbf{0}_{m\times n} & (B_2^T)^{-1} \\ E_{m\times m} & -\Theta^T \end{pmatrix}. \tag{33}$$

As $\Omega = \Omega_1 B_0^{-1}$, from (33) for any integer vector $\boldsymbol{z}$ we have

$$\Omega\boldsymbol{z} = \Omega\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix} = \Omega_1\begin{pmatrix}(B_2^T)^{-1}\boldsymbol{y} \\ \boldsymbol{x}-\Theta^T\boldsymbol{y}\end{pmatrix}.$$

The product

$$\Pi(\boldsymbol{y}) = \prod_{\nu=1}^{d}\left(\sum_{j=1}^{d}\omega_j^{(\nu)}w_j\right)$$

is a symmetric function in each system $\omega_j^{(1)},...,\omega_j^{(d)}$, $1 \leq j \leq d$. Hence it is a rational number with a bounded denominator. So there exists an effectve positive $\varkappa_1 = \varkappa_1(\Omega)$ such that

$$|\Pi(\boldsymbol{z})| \geq \varkappa_1$$

for any nonzero integer vector $\boldsymbol{z}$. From the structure of the matrix Ω_1 (see (32)), we have $\Pi(\boldsymbol{z}) =$

$$\prod_{\nu=1}^{n}\left(\sum_{j=1}^{n}\omega_j^{(\nu)}\left(x_j - \sum_{i=1}^{n}\theta_{i,j}y_i\right) + \sum_{j=1}^{n}\hat{\beta}_{i,j}y_j\right) \times \prod_{\nu=1}^{m}\left(\sum_{j=1}^{n}\omega_j^{(n+\nu)}\left(x_j - \sum_{i=1}^{n}\theta_{i,j}y_i\right)\right),$$

where $\hat{\beta}_{i,j}$ are the entries of the matrix $(B_2^T)^{-1}$. So

$$\max_{1\leq j\leq m}\left|\left|\sum_{i=1}^{n}\theta_{i,j}y_i\right|\right|^m \cdot \max_{1\leq i\leq n}|y_i|^n \geq \gamma_1$$

for all nonzero $\boldsymbol{y} \in \mathbb{Z}^n$ with an effective $\gamma_1 = \gamma_1(\boldsymbol{\beta}) > 0$ and the matrix Θ^T is badly approximable.

4. Additional thoughts on algebraic transference. In this section we show that the statement about bad approximability of the transposed matrix Θ^T may be obtained directly by a Liouville-type argument.

4.1. Matrix relations. We again consider the dual bases $\beta_1, \ldots, \beta_d$ $\omega_1, \ldots, \omega_d$ of the field $\mathbb{K}$, so

$$\sum_{\nu=1}^{d}\beta_i^{(\nu)}\omega_j^{(\nu)} = \delta_{ij}.$$

Then for the matrices

$$B = \begin{pmatrix} \beta_1^{(1)} & \cdots & \beta_m^{(1)} & \beta_{m+1}^{(1)} & \cdots & \beta_d^{(1)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \beta_1^{(n)} & \cdots & \beta_m^{(n)} & \beta_{m+1}^{(n)} & \cdots & \beta_d^{(n)} \\ \beta_1^{(n+1)} & \cdots & \beta_m^{(n+1)} & \beta_{m+1}^{(n+1)} & \cdots & \beta_d^{(n+1)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \beta_1^{(d)} & \cdots & \beta_m^{(d)} & \beta_{m+1}^{(d)} & \cdots & \beta_d^{(d)} \end{pmatrix} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix},$$

$$W = \begin{pmatrix} \omega_1^{(1)} & \cdots & \omega_m^{(1)} & \omega_{m+1}^{(1)} & \cdots & \omega_d^{(1)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \omega_1^{(n)} & \cdots & \omega_m^{(n)} & \omega_{m+1}^{(n)} & \cdots & \omega_d^{(n)} \\ \omega_1^{(n+1)} & \cdots & \omega_m^{(n+1)} & \omega_{m+1}^{(n+1)} & \cdots & \omega_d^{(n+1)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \omega_1^{(d)} & \cdots & \omega_m^{(d)} & \omega_{m+1}^{(d)} & \cdots & \omega_d^{(d)} \end{pmatrix} = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix}$$

we have

$$B^TW = WB^T = I_d.$$

This gives eight relations on B_i and W_i. From these relations we need the following ones:

$$W_3B_1^T + W_4B_2^T = \mathbf{0}_{m\times n}.$$

In the case when the matrices B_2 and W_3 are invertible, we have the following equivalent relation:

$$W_3^{-1}W_4 = -B_1^T(B_2^T)^{-1}. \tag{34}$$

This equality means that the first n rows of B are orthogonal to the last m rows of W.

4.2. More on algebraic transference. Put $\Theta = B_2^{-1}B_1$, then $W_3^{-1}W_4 = -\Theta^T$ (see (34)). Now the bad approximability of Θ and Θ^T can be deduced from a Liouville-type argument for B and W, as we have equalities

$$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = \begin{pmatrix} B_2 & 0_{n\times m} \\ 0_{m\times n} & I_m \end{pmatrix} \begin{pmatrix} \Theta & I_n \\ B_3 & B_4 \end{pmatrix}, \tag{35}$$

$$W = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix} = \begin{pmatrix} I_n & 0_{n\times m} \\ 0_{m\times n} & W_3 \end{pmatrix} \begin{pmatrix} W_1 & W_2 \\ I_m & -\Theta^T \end{pmatrix}. \tag{36}$$

For both B and W, it follows from (35) and (36) that for all nonzero $\boldsymbol{x} \in \mathbb{Z}^m$, $\boldsymbol{y} \in \mathbb{Z}^n$ we have

$$\|\Theta\boldsymbol{x}\|^n|\boldsymbol{x}|^m \gg_B 1,$$
$$\|\Theta^T\boldsymbol{y}\|^m|\boldsymbol{y}|^n \gg_W 1.$$

In Section 3 we used the identity

$$W = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix} = \begin{pmatrix} I_n & W_1 \\ 0_{m\times n} & W_3 \end{pmatrix} \begin{pmatrix} 0_{n\times m} & (B_2^T)^{-1} \\ I_m & -\Theta^T \end{pmatrix}.$$

In fact one may use

$$W = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix} = \begin{pmatrix} * & * \\ 0_{m\times n} & W_3 \end{pmatrix} \begin{pmatrix} * & * \\ I_m & W_3^{-1}W_4 \end{pmatrix},$$

with the same result. Identities (35) and (36) may be extended to

$$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = \begin{pmatrix} B_2 & 0_{n\times m} \\ B_4 & B_3 - B_4\Theta \end{pmatrix} \begin{pmatrix} \Theta & I_n \\ I_m & 0_{m\times n} \end{pmatrix}, \tag{37}$$

$$W = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix} = \begin{pmatrix} W_2 + W_1\Theta^T & W_1 \\ 0_{m\times n} & W_3 \end{pmatrix} \begin{pmatrix} 0_{n\times m} & I_n \\ I_m & -\Theta^T \end{pmatrix}. \tag{38}$$

5. Example. Let α be a totally real algebraic number of degree 4. Then $1, \alpha, \alpha^2, \alpha^3$ form a basis of the totally real field $\mathbb{K} = \mathbb{Q}(\alpha)$. Let $\alpha' \neq \alpha$ be a conjugate of α. Then we may take

$$B_2^T = \begin{pmatrix} 1 & 1 \\ \alpha & \alpha' \end{pmatrix}, \quad B_1^T = \begin{pmatrix} \alpha^2 & \alpha'^2 \\ \alpha^3 & \alpha'^3 \end{pmatrix} = B_2^T \times \begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha'^2 \end{pmatrix}.$$

We see that the matrix

$$M = \frac{1}{\alpha' - \alpha} \begin{pmatrix} \alpha^2 & \alpha'^2 \\ \alpha^3 & \alpha'^3 \end{pmatrix} \begin{pmatrix} \alpha' & -1 \\ -\alpha & 1 \end{pmatrix} = \begin{pmatrix} -\alpha\alpha' & \alpha + \alpha' \\ -\alpha\alpha'(\alpha + \alpha') & \alpha^2 + \alpha'^2 + \alpha\alpha' \end{pmatrix}$$

is badly approximable.

Now we deal with a particular example. Let us consider $\alpha = \sqrt{2+\sqrt{2}}$ that is a root of the irreducible polynomial $(x^2-2)^2-2$. Its conjugates are

$$\pm\sqrt{2\pm\sqrt{2}}.$$

Here we discuss two choices of α'.

The **first** choice is $\alpha' = -\sqrt{2+\sqrt{2}}$. In this case we obtain the matrix

$$M = \begin{pmatrix} 2+\sqrt{2} & 0 \\ 0 & 2+\sqrt{2} \end{pmatrix}.$$

The corresponding system of two linear forms splits into two independent forms

$$(2+\sqrt{2})x_1 - y_1, \;\; (2+\sqrt{2})x_2 - y_2$$

which come from a single quadratic irrationality, and so it is badly approximable **trivially**.

The **second** choice is $\alpha' = \sqrt{2-\sqrt{2}}$. In this case we obtain a **non-trivial** badly approximable matrix

$$M = \begin{pmatrix} -\sqrt{2} & \sqrt{4+2\sqrt{2}} \\ -2\sqrt{2+\sqrt{2}} & 4+\sqrt{2} \end{pmatrix}.$$

6. A result on existence of a basis in a subspace. Analyzing the examples of badly approximable matrices corresponding to the totally real fields of degree 4, we found an interesting property of bases of algebraic fields which form vectors from certain subspaces. Here we formulate an easy theorem which deals with a two-dimensional subspace of a four-dimensional space. It is very likely that a result like this theorem can be obtained in a more general situation.

We consider the case when θ is a real algebraic number of degree 4. We deal with the algebraic field $\mathbb{K} = \mathbb{Q}(\theta)$ and the vector space $\mathfrak{K} = \mathbb{K}^4$. Let $\mathfrak{L}$ be a two-dimensional linear subspace of $\mathfrak{K}$.

Theorem. *The following two statements are equivalent:*
(A) $\forall \boldsymbol{x} \in \mathfrak{L}\;\; \exists \boldsymbol{z} \in \mathbb{Z}^4 \setminus \{\mathbf{0}\}$ *such that* $\boldsymbol{x} \perp \boldsymbol{z}$
(B) $\exists \boldsymbol{z} \in \mathbb{Z}^4 \setminus \{\mathbf{0}\}$ *such that* $\forall \boldsymbol{x} \in \mathfrak{L}: \;\; \boldsymbol{x} \perp \boldsymbol{z}$.

This theorem has an equivalent formulation: *given* $\mathfrak{L}$, *either there exists an integer vector* $\boldsymbol{z} \in \mathbb{Z}^4 \setminus \{\mathbf{0}\}$ *such that* $\boldsymbol{z} \perp \mathfrak{L}$ *or there exists a vector* $\boldsymbol{\omega} = (\omega_0, \omega_1, \omega_2, \omega_3) \in \mathfrak{K}$ *such that the numbers* $\omega_0, \omega_1, \omega_2, \omega_3$ *form a basis for the field* $\mathbb{K}$.

Proof. We may suppose that the two-dimensional subspace $\mathfrak{L}$ has a basis $\boldsymbol{\alpha}, \boldsymbol{\beta}$ written in coordinates in $\mathfrak{K} = \mathbb{K}^4$ as

$$\boldsymbol{\alpha} = (1, 0, \alpha_1, \alpha_2), \quad \boldsymbol{\beta} = (0, 1, \beta_1, \beta_2), \qquad \alpha_j, \beta_j \in \mathbb{K}.$$

We consider several cases.

Case 1. Either numbers $1, \alpha_1, \alpha_2$ or numbers $1, \beta_1, \beta_2$ are linearly independent over $\mathbb{Z}$. Suppose that $1, \alpha_1, \alpha_2$ are independent, and consider the vector

$$\boldsymbol{\alpha} + \xi\boldsymbol{\beta} = (1, \xi, \alpha_1 + \beta_1\xi, \alpha_1 + \beta_2\xi) \in \mathfrak{L}.$$

We consider $\mathbb{K}$ as a vector space over $\mathbb{Q}$. We fix a basis in this vector space and suppose that that with respect to this basis elements α_j, ξ have coordinates

$$\alpha_j = (a_{0,j}, a_{1,j}, a_{2,j}, a_{3,j}), \;\; j = 1, 2, \quad \xi = (x_0, x_1, x_2, x_3).$$

Let the operator of multiplication by β_j in $\mathbb{K}$ be defined by

$$\xi = (x_0, x_1, x_2, x_3) \mapsto (L_{0,j}(\xi), L_{1,j}(\xi), L_{2,j}(\xi), L_{3,j}(\xi)),$$

where $L_j(\xi)$ are homogeneous linear forms in x_0, x_1, x_2, x_3. We consider the determinant

$$\Delta(\xi) = \begin{vmatrix} 1 & x_0 & a_{0,1} + L_{0,1}(\xi) & a_{0,2} + L_{0,2}(\xi) \\ 0 & x_1 & a_{1,1} + L_{1,1}(\xi) & a_{1,2} + L_{1,2}(\xi) \\ 0 & x_2 & a_{2,1} + L_{2,1}(\xi) & a_{2,2} + L_{2,2}(\xi) \\ 0 & x_2 & a_{3,1} + L_{3,1}(\xi) & a_{3,2} + L_{3,2}(\xi) \end{vmatrix} = \begin{vmatrix} x_1 & a_{1,1} + L_{1,1}(\xi) & a_{1,2} + L_{1,2}(\xi) \\ x_2 & a_{2,1} + L_{2,1}(\xi) & a_{2,2} + L_{2,2}(\xi) \\ x_2 & a_{3,1} + L_{3,1}(\xi) & a_{3,2} + L_{3,2}(\xi) \end{vmatrix}.$$

As $1, \alpha_1, \alpha_2$ are independent over $\mathbb{Z}$, the triples

$$a_{1,1}, a_{2,1}, a_{3,1} \;\text{ and }\; a_{1,2}, a_{2,2}, a_{3,2}$$

are not proportional. So the linear part

$$\begin{vmatrix} x_1 & a_{1,1} & a_{1,2} \\ x_2 & a_{2,1} & a_{2,2} \\ x_2 & a_{3,1} & a_{3,2} \end{vmatrix}$$

of the function $\Delta(\xi)$ is not equal to zero identically. This means that there exists ξ_0 with $\Delta(\xi_0) \neq 0$. So the coordinates of the vector $\boldsymbol{\alpha} + \xi_0 \boldsymbol{\beta} \in \mathfrak{L}$ form a basis for $\mathbb{K}$.

Case 2. Both triples $1, \alpha_1, \alpha_2$ and $1, \beta_1, \beta_2$ consist of numbers linearly dependent over $\mathbb{Z}$. We consider several sub-cases.

Case 2.1. There exists j such that α_j, β_j are both rational. Suppose that $j = 1$, without loss of generality. Then $\alpha_1 = \frac{A}{Q}, \; \beta_1 = \frac{B}{Q}, \; \gcd(A, B, Q) = 1$ and the vector $\boldsymbol{n} = (A, B, -Q, 0) \in \mathbb{Z}^4 \setminus \{\boldsymbol{0}\}$ is orthogonal to both, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. So $\boldsymbol{n} \perp \mathfrak{L}$.

Case 2.2. Either α_1, α_2 or β_1, β_2 are both rational. If α_1, α_2 are rational, the set $\{\boldsymbol{z} \in \mathbb{Z}^4 : \; \boldsymbol{z} \perp \boldsymbol{\alpha}\}$ forms a three-dimensional lattice Λ_3. As $1, \beta_1, \beta_2$ are dependent, the set $\{\boldsymbol{z} \in \mathbb{Z}^4 : \; \boldsymbol{z} \perp \boldsymbol{\beta}\}$ forms a two-dimensional lattice Λ_2. It is clear that the intersection $\Lambda_3 \cap \Lambda_2$ contains a nonzero integer vector $\boldsymbol{w}$ and this vector is orthogonal to the whole subspace $\mathfrak{L}$.

Case 2.3. Either α_1, β_2 or α_2, β_1 are both irrational. Without loss of generality we suppose that $\alpha_1, \beta_2 \notin \mathbb{Q}$. Then there exist rationals $a, b, c, d \in \mathbb{Q}$ such that

$$\boldsymbol{\alpha} = (1, 0, \alpha, a + b\alpha), \quad \boldsymbol{\beta} = (0, 1, c + d\beta, \beta),$$

with $\alpha, \beta \in \mathbb{K} \setminus \mathbb{Q}$.

Case 2.3.1. $bd = 1$. Here we consider the vector

$$\boldsymbol{w} = (-a, bc, -b, 1) \in \mathbb{Z}^4.$$

It is clear that $\boldsymbol{w} \perp \boldsymbol{\alpha}, \boldsymbol{w} \perp \boldsymbol{\beta}$ and so $\boldsymbol{w} \perp \mathfrak{L}$.

Case 2.3.2. $bd \neq 1$. Then consider four numbers

$$1, \xi, \alpha + (c + d\beta)\xi, a + b\alpha + \beta\xi \tag{39}$$

which are coordinates of the vector $\boldsymbol{\alpha} + \xi\boldsymbol{\beta} \in \mathfrak{L}$. These numbers form a basis of $\mathbb{K}$ simultaneously with the numbers

$$1, \xi, \alpha, \beta\xi. \tag{40}$$

Case 2.3.2.1. Both irrational numbers $\alpha, \beta \in \mathbb{K}$ are algebraic numbers of degree 2. Then we may suppose that $\alpha = \sqrt{p}$, $\beta = \sqrt{q}$ with squarefree p and q.

Case 2.3.2.1.1. $p = q$. Then take $\xi \in \mathbb{K}$ such that $\mathbb{K} = (\mathbb{Q}(\sqrt{q}))(\xi)$. Then numbers (40) form a basis for $\mathbb{K}$ over $\mathbb{Q}$. So numbers (39) form a basis for $\mathbb{K}$ also.

Case 2.3.2.1.2. $p \neq q$. Then the numbers

$$1, \sqrt{p}, \sqrt{q}, \sqrt{pq}$$

form a basis of $\mathbb{K}$ over $\mathbb{Q}$. Then numbers (40) form a basis for $\mathbb{K}$ over $\mathbb{Q}$ with $\xi = \sqrt{p} + \sqrt{q}$.

Case 2.3.2.2. At least one of the irrational numbers $\alpha, \beta \in \mathbb{K}$ has degree 4. Without loss of generality we suppose that $\deg \beta = 4$. Then $1, \beta, \beta^2, \beta^3$ is a basis for $\mathbb{K}$. Suppose that

$$\alpha = a_0 + a_1\beta + a_2\beta^2 + a_3\beta^3, \quad \xi = x_0 + x_1\beta + x_2\beta^2 + x_3\beta^3$$

with rational a_j, x_j. Let

$$z^4 + B_3z^3 + B_2z^2 + B_1z + B_0, \quad B_j \in \mathbb{Q}$$

be the minimal polynomial for β. Then

$$\beta\xi = -B_0x_3 + (x_0 - B_1x_3)\beta + (x_1 - B_2x_3)\beta^2 + (x_2 - B_3x_3)\beta^3.$$

Consider the determinant

$$D(\xi) = \begin{vmatrix} 1 & x_0 & a_0 & -B_0x_3 \\ 0 & x_1 & a_1 & x_0 - B_1x_3 \\ 0 & x_2 & a_2 & x_1 - B_2x_3 \\ 0 & x_3 & a_3 & x_2 - B_3x_3 \end{vmatrix} = \begin{vmatrix} x_1 & a_1 & x_0 - B_1x_3 \\ x_2 & a_2 & x_1 - B_2x_3 \\ x_3 & a_3 & x_2 - B_3x_3 \end{vmatrix}.$$

As $\alpha \notin \mathbb{Q}$, at least one of the monomials $-a_1x_2^2, -a_2x_0x_3, -a_3x_1^2$ of $D(\xi)$ does not vanish. So there exists $\xi_0 \in \mathbb{K}$ such that $D(\xi_0) \neq 0$. Then numbers (40) form a basis for $\mathbb{K}$. $\square$

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