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ON LATTICES GENERATED BY FINITE ABELIAN GROUPS*

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Abstract. This paper is devoted to the study of lattices generated by finite Abelian groups. Special species of such lattices arise in the exploration of elliptic curves over finite fields. In the case where the generating group is cyclic, they are also known as the Barnes lattices. It is shown that for every finite Abelian group with the exception of the cyclic group of order four these lattices have a basis of minimal vectors. Another result provides an improvement of a recent upper bound by M. Sha for the covering radius in the case of the Barnes lattices. Also discussed are properties of the automorphism groups of these lattices.

Key words. well-rounded lattice, finite Abelian group, minimal vector, covering radius, automorphism group, Toeplitz determinant

AMS subject classifications. Primary, 11H31; Secondary, 11G20, 11H55, 15A15, 15B05, 52C17

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1. Introduction.

The lattice generated by a finite Abelian (additive) group $G = \{0, g_1, \dots, g_n\}$ of order $|G| = n+1$ is defined as

$$\mathcal{L}(G) := \{X = (x_1, \dots, x_n, -x_1 - \cdots - x_n) \in \mathbf{Z}^{n+1} : x_1 g_1 + \cdots + x_n g_n = 0\}.$$

We think of this lattice as a sublattice of full rank n of the root lattice

$$\mathcal{A}_n := \{(x_1, \dots, x_n, -x_1 - \cdots - x_n) \in \mathbf{Z}^{n+1}\}.$$

We denote by $d(G)$ the minimum distance in $\mathcal{L}(G)$, that is, with $\|\cdot\|$ denoting the Euclidean norm,

$$d(G) := \min\{\|X\| : X \in \mathcal{L}(G) \setminus \{0\}\},$$

and we let $\mathcal{S}(G)$ stand for the set of nonzero lattice vectors of minimal length, that is, for the set of all $X \in \mathcal{L}(G)$ with $\|X\| = d(G)$. The lattice $\mathcal{L}(G)$ is said

- to be well-rounded if $\mathcal{S}(G)$ contains n linearly independent vectors,
- to be generated by minimal vectors if $\operatorname{span}_{\mathbf{Z}} \mathcal{S}(G) = \mathcal{L}(G)$, that is, if each vector in $\mathcal{L}(G)$ is a linear combination with integer coefficients of vectors in $\mathcal{S}(G)$,
- to have a basis of minimal vectors if $\mathcal{S}(G)$ contains n vectors such that each lattice vector is a linear combination with integer coefficients of these n vectors.

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Clearly, each of these properties implies its predecessor. Lattices in Euclidean spaces satisfying any of the above properties are of importance in extremal lattice theory, discrete geometry, and combinatorics. Such lattices usually have a high degree of symmetry, which allows for some classical discrete optimization problems to be reduced to them (see [17] for detailed information). It is especially interesting when lattices with these properties come from algebraic constructions, hence inheriting additional algebraic structure. For instance, there are well-known lattice constructions from ideals in number fields [2], [3], ideals in polynomial rings [16], and curves over finite fields [22, pp. 578–583]. In addition to their intrinsic theoretical value, such lattices also have many applications, for instance in coding theory and cryptography, as described in [22] and [16], respectively.

Our present construction of lattices from Abelian groups generalizes the special case of a family of lattices coming from elliptic curves over finite fields as in [22], which has recently been investigated in [10] and [20]. It is our goal to show that these lattices have some remarkable geometric properties, including those listed above. Here is our first observation. We abbreviate $\mathbf{Z}/n\mathbf{Z}$ to $\mathbf{Z}_n$.

THEOREM 1.1. *Except for the lattice $\mathcal{L}(\mathbf{Z}_4)$, which is not well-rounded, the lattice $\mathcal{L}(G)$ is well-rounded for every finite Abelian group G. The minimum distance is $\sqrt{8}$ for $G = \mathbf{Z}_2$, is $\sqrt{6}$ for $G = \mathbf{Z}_3$, and is equal to $\sqrt{4} = 2$ for all other finite Abelian groups G.*

Our first main result is as follows.

THEOREM 1.2. *For every finite Abelian group $G \neq \mathbf{Z}_4$, the lattice $\mathcal{L}(G)$ has a basis of minimal vectors.*

Theorem 1.1 implies that $\mathcal{L}(\mathbf{Z}_4)$ does not possess a basis of minimal vectors. Of course, Theorem 1.2 is stronger than Theorem 1.1. We nevertheless give an independent proof for Theorem 1.1, because well-roundedness may be proved by arguments that are much simpler than those we have to invoke to establish Theorem 1.2.

One of the subtleties of lattices, discovered by Conway and Sloane [8], is that a lattice generated by minimal vectors need not have a basis of minimal vectors. More recently, it has been shown [18] that this phenomenon takes place for some lattices in dimensions ≥ 10, but not in lower dimensions. Theorem 1.2 implies that this does not happen for the class of lattices explored in this paper.

The lattices we study here include the lattices which come from elliptic curves over finite fields, namely, $\mathcal{L}(G)$, where G is the group of rational points on an elliptic curve over a finite field. These groups were completely described by Rück [19], and they are always of the form $G = \mathbf{Z}_{m_1} \times \mathbf{Z}_{m_2}$, the direct product of two cyclic groups (with further restrictions on the possible values of (m_1, m_2)). For lattices coming from elliptic curves over finite fields, paper [10] contains Theorem 1.1 and the weaker version of Theorem 1.2 which states that for $|G| \geq 5$ the lattice $\mathcal{L}(G)$ is generated by minimal vectors, while Sha [20] proved Theorem 1.2 for those lattices. The contribution of the present paper is that we extend these results to general finite Abelian groups G.

Well-rounded lattices play a crucial role in the theory of sphere packing (see [9], [17]), where maximal nonoverlapping balls of equal radius (equal to half of the minimal distance of the lattice) are centered at the lattice points with the goal of covering the largest possible proportion of the ambient space. This proportion, called the packing density of the lattice, is equal to the volume of one such ball divided by the volume of a fundamental domain of the lattice (equal to the determinant of the lattice). The lattice packing problem consists in finding a lattice of prescribed dimension whose packing density is maximal. This emphasizes the importance of knowing the minimal distance and the determinant of the lattice.

Our second topic of investigation is related to another classical optimization problem on lattices, the sphere covering problem (again, see [9], [17]). The goal is to cover the ambient space completely by balls of equal radius (called the covering radius of the lattice) centered at the lattice points, minimizing the proportion of overlap of these balls. A variety of classical general bounds for covering radii of lattices (also referred to as inhomogeneous minima) can be found in [13, Chap. 2, section 13]. Here we present estimates for the covering radius $\mu(G)$ of $\mathcal{L}(G)$. By definition, $\mu(G)$ is the smallest number μ such that

$$\operatorname{span}_{\mathbf{R}}\mathcal{A}_n := \{(\xi_1,\dots,\xi_n,-\xi_1-\cdots-\xi_n)\in\mathbf{R}^{n+1} : \xi_1,\dots,\xi_n\in\mathbf{R}\}$$

is covered by n-dimensional closed Euclidean balls in $\operatorname{span}_{\mathbf{R}}\mathcal{A}_n$ of radius μ centered at the points of $\mathcal{L}(G)$. The covering radii for the small groups are

$$\begin{aligned}
&\mu(\mathbf{Z}_2) = \sqrt{2} \approx 1.4142,\\
&\mu(\mathbf{Z}_3) = \sqrt{2} \approx 1.4142,\\
&\mu(\mathbf{Z}_4) = \frac{3}{2} = 1.5000, \quad \mu(\mathbf{Z}_2\times\mathbf{Z}_2) = \sqrt{3} \approx 1.7321,\\
&\mu(\mathbf{Z}_5) = \sqrt{2} \approx 1.4142,\\
&\mu(\mathbf{Z}_6) = \sqrt{\frac{17}{8}} \approx 1.4577.
\end{aligned}$$

For general finite Abelian groups G, we obviously have $\mu(G)\geq\mu(\mathcal{A}_n)$, where $\mu(\mathcal{A}_n)$ is the covering radius of $\mathcal{A}_n$, which is known to be

$$\mu(\mathcal{A}_n) = \begin{cases} \frac{1}{2}\sqrt{n+1} & \text{if } n \text{ is odd},\\ \frac{1}{2}\sqrt{n+1-1/(n+1)} & \text{if } n \text{ is even}; \end{cases}$$

see [9, Chap. 4, section 6.1]. In [20], it is shown that $\mu(G)\leq\mu(\mathcal{A}_n)+\sqrt{2}$. This is a significant improvement of the estimate $\mu(G)\leq n$, which, for $n\geq 5$, follows from Jarnik's classical bound via successive minima [15] along with the fact that G is well-rounded for $n\geq 5$ due to Theorem 1.1.

If $G=\mathbf{Z}_{n+1}$ is the cyclic group of the numbers $0,1,\dots,n$ with addition modulo $n+1$, then $\mathcal{L}(G)$ is the sublattice of $\mathcal{A}_n$ formed by the points satisfying

$$x_1+2x_2+\cdots+nx_n = 0 \ \text{ modulo } \ n+1.$$

These lattices probably first appeared in [1] and are therefore frequently referred to as the Barnes lattices. Here is another main result of this paper. It provides us with an improvement of the upper bound $\mu(\mathcal{A}_n)+\sqrt{2}$ for cyclic groups, that is, for the Barnes lattices.

THEOREM 1.3. *For every $n\geq 2$,*

$$\mu(\mathbf{Z}_{n+1}) < \frac{1}{2}\sqrt{n+4\log(n-1)+7-4\log 2+10/n}.$$

The data (chopped after the fourth digit after the decimal point) for several values of n are shown in Table 1. We remark that, for $n\geq 4$, Theorem 1.3 even holds with $4\log(n-1)-4\log 2$ replaced by the slightly smaller number $4\sum_{k=3}^{n-1}(1/k)$.

Third, we investigate a certain property of the automorphism groups of our lattices $\mathcal{L}(G)$, which is intrinsically related to their algebraic construction. The automorphism group $\operatorname{Aut}(\mathcal{L})$ of a full rank sublattice $\mathcal{L}$ of some lattice $\mathcal{A}$ is defined as the

TABLE 1

n	$\mu(\mathcal{A}_n)$	Theorem 1.3	$\mu(\mathcal{A}_n)+\sqrt{2}$
3	1.0000	1.8257	2.4142
4	1.0954	1.9443	2.5097
5	1.2247	2.0477	2.6390
6	1.3093	2.1408	2.7235
20	2.2887	3.0210	3.7029
50	3.5700	4.1831	4.9842
100	5.0247	5.5387	6.4389
1 000	15.8193	16.0613	17.2335
10 000	50.0025	50.1026	51.4167
100 000	158.1147	158.1536	159.5289
1 000 000	500.0002	500.0149	501.4145

group of all maps of $\mathcal{L}$ onto itself which extend to linear isometries of $\operatorname{span}_{\mathbf{R}}\mathcal{A}$. It is easily seen that in our setting, $\mathcal{L}(G)\subset\mathcal{A}_n$, a map $\tau\in\operatorname{Aut}(\mathcal{L}(G))$ is necessarily of the form

$$\tau(X)=\tau\left(x_1,\dots,x_n,-\sum_{i=1}^n x_i\right)=\left(Ux,-\sum_{i=1}^n (Ux)_i\right)$$

with some matrix $U\in\mathrm{GL}_n(\mathbf{Z})$. Here $x=(x_1,\dots,x_n)^\top$. We therefore identify $\operatorname{Aut}(\mathcal{L}(G))$ with a subgroup of $\mathrm{GL}_n(\mathbf{Z})$. It is a well-known fact that any finite subgroup of $\mathrm{GL}_n(\mathbf{Z})$ is the automorphism group of some lattice. In all dimensions except for $n=2,4,6,7,8,9,10$ (dimensions with exceptionally symmetric lattices) the largest such group is $(\mathbf{Z}/2\mathbf{Z})^n\rtimes S_n$, the automorphism group of the integer lattice $\mathbf{Z}^n$; here S_n is the symmetric group on n letters viewed as the subgroup of $\mathrm{GL}_n(\mathbf{Z})$ consisting of the permutation matrices (see [9], [17], and [21] for more information on automorphism groups of lattices). Lattices with large automorphism groups usually have a large degree of geometric symmetry, which often correlates with having many minimal vectors and well-roundedness. In particular, the relation between certain properties of $\operatorname{Aut}(\mathcal{L})\cap S_n$ and the probability of $\mathcal{L}$ being well-rounded has recently been investigated in [11], [12]. Here we prove the following.

THEOREM 1.4. *For every finite Abelian group G,*

$$\operatorname{Aut}(\mathcal{L}(G))\cap S_n\cong\operatorname{Aut}(G),$$

where $\operatorname{Aut}(G)$ *is the group of automorphisms of* G.

This result, along with a characterization of the automorphism groups of finite Abelian groups, for which see, e.g., [14], helps to understand the symmetries of our family of lattices $\mathcal{L}(G)$.

The paper is organized as follows. Section 2 is devoted to the determinant of $\mathcal{L}(G)$. There we first present a short derivation based on a general fact from lattice theory and then give a second proof, which uses only elementary facts for determinants, mainly the Cauchy–Binet formula. The proofs of Theorems 1.1 and 1.2 we give here occupy sections 3 to 6 and use tools from linear algebra only. Again the Cauchy–Binet formula is always the key. Theorem 1.3 is proved in section 7. The proof is anew pure linear algebra and makes use of explicit formulas for certain Toeplitz determinants. Finally, in section 8 we prove Theorem 1.4 and comment on a certain geometric interpretation of this result.

2. The determinant. A set of n vectors $X_1,\dots,X_n\in\mathcal{L}(G)$ is called a basis if each vector in $\mathcal{L}(G)$ is a linear combination with integer coefficients of these vectors.

In that case the parallelotope spanned by $X_1, \dots, X_n$ is referred to as a fundamental parallelotope. All fundamental parallelotopes have the same volume. This volume is denoted by $\det \mathcal{L}(G)$ and referred to as the determinant of the lattice $\mathcal{L}(G)$. Even more can be said: the parallelotope spanned by n vectors $X_1, \dots, X_n \in \mathcal{L}(G)$ has the volume $\det \mathcal{L}(G)$ if *and only if* these vectors form a basis of $\mathcal{L}(G)$. If $X_1, \dots, X_n \in \mathcal{L}(G)$ form a basis, then the $(n+1) \times n$ matrix B whose jth column is constituted by the $n+1$ coordinates of X_j is called a basis matrix. If B is an arbitrary basis matrix of $\mathcal{L}(G)$, then $\det \mathcal{L}(G) = \sqrt{\det B^\top B}$, where the determinant on the right is the usual determinant of an $n \times n$ matrix. All these results are standard in lattice theory and can be found in [9], [13], or [17], for example.

It turns out that $\det \mathcal{L}(G) = |G|^{3/2} = (n+1)^{3/2}$. The following proof of this formula is from [20, Proposition 5.1], where it is given for the case when G is a subgroup of the group of rational points on an elliptic curve over a finite field; it also holds verbatim for general Abelian groups G. Let $\mathcal{L}$ be a sublattice of full rank of some lattice $\mathcal{A} \subset \mathbf{R}^N$. Think of $\mathcal{A}$ as an (Abelian) additive group and consider $\mathcal{L}$ as a subgroup of $\mathcal{A}$. A basic result of lattice theory says that if the quotient group $\mathcal{A}/\mathcal{L}$ has finite order $|\mathcal{A}/\mathcal{L}|$, then $\det \mathcal{L}/\det \mathcal{A} = |\mathcal{A}/\mathcal{L}|$. Now take $\mathcal{A} = \mathcal{A}_n$ and $\mathcal{L} = \mathcal{L}(G)$. It is known that $\det \mathcal{A}_n = \sqrt{n+1}$. The group homomorphism

$$\varphi : \mathcal{A}_n \to G, \quad (x_1, \dots, x_n, -x_1 - \cdots - x_n) \mapsto x_1 g_1 + \cdots x_n g_n$$

is surjective and its kernel is just $\mathcal{L}(G)$. Consequently, $\mathcal{A}_n/\mathcal{L}(G)$ is isomorphic to G, which implies that $|\mathcal{A}_n/\mathcal{L}(G)| = |G| = n+1$. It follows that

$$\frac{\det \mathcal{L}(G)}{\sqrt{n+1}} = \frac{\det \mathcal{L}(G)}{\det \mathcal{A}_n} = |\mathcal{A}_n/\mathcal{L}(G)| = n+1,$$

as asserted.

Here is a purely linear algebra proof of the same determinant formula. We first exemplify the idea by considering $G = \mathbf{Z}_2 \times \mathbf{Z}_4$. The lattice $\mathcal{L}(G)$ consists of the points

$$(x_1, x_2, y_{02}, y_{03}, y_{11}, y_{12}, y_{13}, -x_1 - x_2 - y_{02} - y_{03} - y_{11} - y_{12} - y_{13}) \in \mathbf{Z}^8$$

satisfying

$$x_1(1,0) + x_2(0,1) + y_{02}(0,2) + y_{03}(0,3) + y_{11}(1,1) + y_{12}(1,2) + y_{13}(1,3) = (0_2, 0_4),$$

where 0_2 and 0_4 are the zeros in $\mathbf{Z}_2$ and $\mathbf{Z}_4$. We may choose the five numbers y_{jk} arbitrarily, after which x_1 and x_2 are determined uniquely modulo 2 and 4, respectively. Taking $y_{jk} = 1$ and $y_{\alpha,\beta} = 0$ for $(\alpha, \beta) \neq (j, k)$, we get $x_1 + j = 0_2$ and $x_2 + k = 0_4$, that is, $x_1 = -j$ modulo 2 and $x_2 = -k$ modulo 4. Thus, a basis in $\mathcal{L}(G)$ is formed by the five rows

$$(-j, -k, 0, \dots, 0, 1, 0, \dots, 0, j + k - 1),$$

the number 1 being at the (j,k)th position in lexicographic order, and by the two rows

$$(2, 0, 0, 0, 0, 0, 0, -2), \quad (0, 4, 0, 0, 0, 0, 0, -4),$$

which allow us to move x_1 and x_2 within $2\mathbf{Z}$ and $4\mathbf{Z}$. It follows that the matrix $B^\top$ formed by these seven rows,

$$B^\top = \begin{pmatrix} 2 & 0 & & & & & -2 \\ 0 & 4 & & & & & -4 \\ 0 & -2 & 1 & & & & 1 \\ 0 & -3 & & 1 & & & 2 \\ -1 & -1 & & & 1 & & 1 \\ -1 & -2 & & & & 1 & 2 \\ -1 & -3 & & & & & 3 \end{pmatrix},$$

is the transpose of a basis matrix B of the lattice $\mathcal{L}(G)$. Thus, $\det \mathcal{L}(G) = \sqrt{\det B^\top B}$. The Cauchy–Binet formula gives

$$\det B^\top B = (\det B_1)^2 + (\det B_2)^2 + \sum_{j,k} (\det B_{jk})^2 + (\det B_7)^2,$$

where B_1, B_2, B_7 result from $B^\top$ by deleting the columns $1, 2, 7$, respectively, and B_{jk} is the matrix obtained by deleting the column with 1 in the position (j,k). Expanding the determinants of B_1, B_2, B_7 along the five columns with a single 1, we see that the squares of these determinants are

$$\begin{vmatrix} 0 & -2 \\ 4 & -4 \end{vmatrix}^2 = 8^2, \quad \begin{vmatrix} 2 & -2 \\ 0 & -4 \end{vmatrix}^2 = 8^2, \quad \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix}^2 = 8^2,$$

and expanding the five determinants $\det B_{jk}$ along their four columns with a single 1 we get

$$(\det B_{jk})^2 = \begin{vmatrix} 2 & 0 & -2 \\ 0 & 4 & -4 \\ -j & -k & j+k-1 \end{vmatrix}^2 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ -j & -k & -1 \end{vmatrix}^2 = 8^2.$$

Thus, $\det B^\top B = 8 \cdot 8^2 = 8^3 = |G|^3$, as desired.

It is clear how to proceed in the general case $G = \mathbf{Z}_{m_1} \times \cdots \times \mathbf{Z}_{m_k}$. Put $m = m_1 \cdots m_k$. Then $B^\top$ has $m-1$ rows and m columns and we may employ the Cauchy–Binet formula to express $\det B^\top B$ as the sum of m squares of determinants as above. The first k and the last squared determinants are readily seen to be $(m_1 \cdots m_k)^2 = m^2$. The $m-k-1$ squared determinants corresponding to indices $(j_1, \ldots, j_k)$ are, with $\sigma := j_1 + \cdots + j_k$,

$$\begin{vmatrix} m_1 & & & & -m_1 \\ & m_2 & & & -m_2 \\ & & \ddots & & \vdots \\ & & & m_k & -m_k \\ -j_1 & -j_2 & \ldots & -j_k & \sigma - 1 \end{vmatrix}^2 = \begin{vmatrix} m_1 & & & & 0 \\ & m_2 & & & 0 \\ & & \ddots & & \vdots \\ & & & m_k & 0 \\ -j_1 & -j_2 & \ldots & -j_k & -1 \end{vmatrix}^2,$$

which equals $(m_1 \cdots m_k)^2 = m^2$. Consequently, $\det B^\top B = m \cdot m^2 = m^3 = |G|^3$.

3. The small groups. We now turn to the proof of Theorems 1.1 and 1.2. In this section we introduce some notation and consider a few examples. The examples will also be used in the proofs in the forthcoming sections.

We arrange the nonzero elements of G in a column $\mathbf{g} = (g_1, \dots, g_n)^\top$ of size n. Obviously, there are $n!$ possibilities to do this. Then each point

$$X = (x_1, \dots, x_n, -(x_1 + \cdots + x_n)) \in \mathcal{L}(G)$$

may be represented by a column $\mathbf{x} = (x_1, \dots, x_n)^\top$ of the same height n. Given n points $X_1, \dots, X_n$ in $\mathcal{L}(G)$, we denote by M the $n \times n$ matrix composed of the columns $\mathbf{x}_1, \dots, \mathbf{x}_n$, and we collect the data in an array $\mathbf{g}||M$. The arrays that may be obtained in this way will be called admissible for G. We let $\widetilde{M}$ stand for the $(n+1) \times n$ matrix which results from adding the row consisting of the negatives of the column sums of the matrix M. Thus, n vectors $X_1, \dots, X_n$ form a basis in $\mathcal{L}(G)$ if and only if $\widetilde{M}$ is a basis matrix, which, because $|G|^3 = (\det \mathcal{L}(G))^2$, is equivalent to the equality $\det \widetilde{M}^\top \widetilde{M} = |G|^3$.

Clearly, n vectors $X_1, \dots, X_n$ are linearly independent if and only if so are the n columns $\mathbf{x}_1, \dots, \mathbf{x}_n$. Therefore, in order to prove that $\mathcal{L}(G)$ is well-rounded, we have to find an admissible array $\mathbf{g}||M$ in which the matrix M is nonsingular and comes from points of minimum distance. To prove the stronger property that $\mathcal{L}(G)$ has a basis of minimal vectors, we have to find an admissible array $\mathbf{g}||M$ associated with points of minimum distance such that $\widetilde{M}$ is a basis matrix.

The minimum distance is always at least $\sqrt{1^2+1^2+1^2+1^2} = 2$ and this distance is attained exactly at the points X containing two times 1, two times -1, and otherwise only zeros. If the lattice does not contain such points, the minimum distance must be at least $\sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$.

Example 3.1. Let G be $\mathbf{Z}_3 = \{0, 1, 2\}$ and let $\mathbf{g} = (1, 2)^\top$. (The other possibility would be to put $\mathbf{g} = (2, 1)^\top$.) Then $\mathcal{L}(\mathbf{Z}_3)$ consists of the integer points $(x, y, -x-y)$ satisfying $x + 2y = 0$ modulo 3. By inspection it is easily seen that $d(\mathbf{Z}_3)$ is $\sqrt{6}$ and that exactly six points of $\mathcal{L}(\mathbf{Z}_3)$ have minimal distance. Two of them are the points $X_1 = (-2, 1, 1)$ and $X_2 = (1, -2, 1)$. The array $\mathbf{g}||M$ corresponding to these two points is

$$\begin{array}{c|cc} 1 & -2 & 1 \\ 2 & 1 & -2 \end{array}.$$

The matrix $M = \left(\begin{smallmatrix} -2 & 1 \\ 1 & -2 \end{smallmatrix}\right)$ is nonsingular, and hence $\mathcal{L}(\mathbf{Z}_3)$ is well-rounded with the minimum distance $\sqrt{6}$. We have

$$\widetilde{M} = \left(\begin{array}{cc} -2 & 1 \\ 1 & -2 \\ \hline 1 & 1 \end{array}\right).$$

Since $\det \widetilde{M}^\top \widetilde{M} = 3^3$, we see that $\widetilde{M}$ is a basis matrix and thus that $\mathcal{L}(\mathbf{Z}_3)$ has a basis of minimal vectors.

Example 3.2. Things are trivial for $G = \mathbf{Z}_2$, in which case $n = 1$. We have $\mathcal{L}(\mathbf{Z}_2) = \{(2x, -2x) : x \in \mathbf{Z}\}$, the minimum distance is $d(\mathbf{Z}_2) = \sqrt{2^2 + 2^2} = \sqrt{8}$, and it is attained for $X = (-2, 2)$ (and also for $X = (2, -2)$).

Example 3.3. Let $G = \mathbf{Z}_4$ and $\mathbf{g} = (1, 2, 3)^\top$. An integer point $(x, y, z, -x-y-z)$ is in $\mathcal{L}(\mathbf{Z}_4)$ if and only if $x + 2y + 3z = 0$ modulo 4. The points of minimum distance are

$$X_1 = (1, 1, -1, -1),\ X_2 = (-1, 1, 1, -1),\ X_3 = (-1, -1, 1, 1),\ X_4 = (1, -1, -1, 1),$$

but any three of them are linearly dependent. Thus, $\mathcal{L}(\mathbf{Z}_4)$ is not well-rounded. Clearly, $d(\mathbf{Z}_4) = 2$.

Example 3.4. For $G = \mathbf{Z}_2 \times \mathbf{Z}_2$, the array

$$\mathbf{g}||M = \begin{array}{c|c} (0,1) & \begin{array}{ccc} 1 & -1 & 1 \end{array} \\ (1,0) & \begin{array}{ccc} 1 & 1 & -1 \end{array} \\ (1,1) & \begin{array}{ccc} -1 & 1 & 1 \end{array} \end{array}$$

is admissible, and since $\det M = 4 \neq 0$, it follows that $\mathcal{L}(\mathbf{Z}_2 \times \mathbf{Z}_2)$ is well-rounded with $d(\mathbf{Z}_2 \times \mathbf{Z}_2) = 2$. The matrix

$$\widetilde{M} = \left(\begin{array}{ccc} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \\ \hline -1 & -1 & -1 \end{array}\right)$$

satisfies $\det \widetilde{M}^\top \widetilde{M} = 4^3$, and hence $\mathcal{L}(\mathbf{Z}_2 \times \mathbf{Z}_2)$ has a basis of minimal vectors.

4. The cyclic groups. Let G be any of the groups $\mathbf{Z}_m$ with $m \geq 5$ and put $\mathbf{g}_m = (1, 2, \ldots, m-1)^\top$. We denote by T_m the $(m-1) \times (m-1)$ tridiagonal Toeplitz matrix with -2 on the main diagonal and 1 on the two neighboring diagonals. For example,

$$T_7 = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}, \quad \widetilde{T}_7 = \left(\begin{array}{cccccc} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & -2 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Let U_m be the $(m-1) \times (m-1)$ matrix which results from the $(m-1) \times (m-1)$ bidiagonal Toeplitz matrix with 1 on the main diagonal and on the subdiagonal after replacing the last column with $(0, \ldots, 0, -1, -1, -1, 0)^\top$. For instance,

$$U_5 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad U_7 = \left(\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

LEMMA 4.1. *Let $M_m = T_m U_m$. Then $\widetilde{M}_m = \widetilde{T}_m U_m$ and the matrix M_m results from the $(m-1) \times (m-1)$ tetradiagonal Toeplitz matrix with first column $(-1, -1, 1, 0, \ldots, 0)^\top$ and first row $(-1, 1, 0, \ldots, 0)$ by replacing the last column with $(1, 0, 1, -1)^\top$ for $m = 5$ and with the column $(0, \ldots, 0, -1, 1, 0, 1, -1)^\top$ for $m \geq 6$.*

Proof. This can be verified by direct computation. □

It can be checked straightforwardly that $\mathbf{g}_m||M_m$ is an admissible array for $\mathbf{Z}_m$. For example, the arrays $\mathbf{g}_5||M_5$ and $\mathbf{g}_7||M_7$ are

$$\mathbf{g}_5||M_5 = \begin{array}{c|cccc} 1 & -1 & 1 & 0 & 1 \\ 2 & -1 & -1 & 1 & 0 \\ 3 & 1 & -1 & -1 & 1 \\ 4 & 0 & 1 & -1 & -1 \end{array}, \quad \mathbf{g}_7||M_7 = \begin{array}{c|cccccc} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & -1 & -1 & 1 & 0 & 0 & -1 \\ 3 & 1 & -1 & -1 & 1 & 0 & 1 \\ 4 & 0 & 1 & -1 & -1 & 1 & 0 \\ 5 & 0 & 0 & 1 & -1 & -1 & 1 \\ 6 & 0 & 0 & 0 & 1 & -1 & -1 \end{array}.$$

It is well known that $\det T_m = (-1)^{m-1}m$. We have $\det U_5 = 1$, which implies that $\det U_m = 1$ for all $m \geq 5$. Consequently, by Lemma 4.1,

$$\det M_m = \det T_m \det U_m = (-1)^{m-1}m \neq 0.$$

This proves that $\mathcal{L}(\mathbf{Z}_m)$ is well-rounded with $d(\mathbf{Z}_m) = 2$. The fact that $\mathcal{L}(\mathbf{Z}_m)$ has a basis of minimal vectors lies a little deeper. It requires the following result from [4]. We include a proof (based on the Cauchy–Binet formula) for the reader's convenience.

LEMMA 4.2. *We have* $\det \widetilde{T}_m^\top \widetilde{T}_m = m^3$.

Proof. Applying the Cauchy–Binet formula, we may write

$$\det \widetilde{T}_m^\top \widetilde{T}_m = (\det C_1)^2 + (\det C_2)^2 + \cdots + (\det C_m)^2,$$

where C_j results from $\widetilde{T}_m$ by deleting the jth row. Clearly, $(\det C_m)^2 = (\det T_m)^2 = m^2$. For $j \leq m-1$, we expand $\det C_j$ along the last row and obtain two block-triangular determinants:

$$\det C_j = (-1)^m \det T_{m-j} + \det T_j = (-1)^m(-1)^{m-j-1}(m-j) + (-1)^{j-1}j = (-1)^{j-1}m.$$

It follows that $(\det C_j)^2 = m^2$. Consequently, $\det \widetilde{T}_m^\top \widetilde{T}_m = m \cdot m^2 = m^3$. ∎

Combining Lemma 4.2 with the factorization $\widetilde{M_m} = \widetilde{T_m} U_m$ delivered by Lemma 4.1, we get $\det \widetilde{M}_m^\top \widetilde{M}_m = m^3$, which shows that $\mathcal{L}(\mathbf{Z}_m)$ is generated by vectors of minimum distance.

5. Direct products: Well-roundedness. In this section we complete the proof of Theorem 1.1. Much of the following, especially the choice of the matrices in the arrays, resembles the constructions in [20]. However, our reasoning is consistently based on the computation of determinants and thus completely differs from the arguments used in [20].

LEMMA 5.1. *If G and H are finite Abelian groups such that $\mathcal{L}(G)$ and $\mathcal{L}(H)$ are well-rounded with $d(G) = d(H) = 2$, then $\mathcal{L}(G\times H)$ is well-rounded and $d(G\times H) = 2$.*

Proof. Let $G = \{0, g_1, \ldots, g_n\}$ and $H = \{0, h_1, \ldots, h_m\}$. We write

$$\mathbf{g} = (g_1, \ldots, g_n)^\top, \quad \mathbf{h} = (h_1, \ldots, h_m)^\top.$$

By assumption, there exist nonsingular integer matrices $M_G = (a_{ij})$ and $M_H = (b_{ij})$ such that $\mathbf{g}||M_G$ and $\mathbf{h}||M_H$ are admissible arrays and such that the columns after deleting all zeros reduce to columns with 3 or 4 entries containing only ± 1 and having

their column sum in $\{-1, 0, 1\}$. The array

$(g_1, 0)$	a_{11}	$\dots$	a_{1n}				-1	-1	
$\vdots$	$\vdots$		$\vdots$						
$(g_n, 0)$	a_{n1}	$\dots$	a_{nn}						$\vdots$
$(0, h_1)$				b_{11}	$\dots$	b_{1m}	-1		
$(0, h_2)$				b_{21}	$\dots$	b_{2m}		-1	
$\vdots$				$\vdots$		$\vdots$			
$(0, h_m)$				b_{m1}	$\dots$	b_{mm}			$\vdots$
(g_1, h_1)							1		
(g_1, h_2)								1	
$\vdots$									$\vdots$

consists of $n + m + nm = (n + 1)(m + 1) - 1$ columns. The last nm columns may be labeled by (g_i, h_j), and the column with this label has 1 at position (g_i, h_j) and -1 at the positions $(g_i, 0)$ and $(0, h_j)$. This array is clearly admissible for $G \times H$, and since its matrix is upper block triangular with determinant $\det M_G \det M_H \neq 0$, we conclude that $\mathcal{L}(G \times H)$ is well-rounded. This array also reveals that the minimal distance of $\mathcal{L}(G \times H)$ is 2, that is, $d(G \times H) = 2$. □

Lemma 5.1 in conjunction with the result of the previous section proves Theorem 1.1 for all groups which do not contain the factors $\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4$.

LEMMA 5.2. *If $m \in \{2, 3, 4\}$ and G is a finite Abelian group such that $\mathcal{L}(G)$ is well-rounded with $d(G) = 2$, then $\mathcal{L}(\mathbf{Z}_m \times G)$ is well-rounded and $d(\mathbf{Z}_m \times G) = 2$.*

Proof. Let $G = \{0, g_1, \dots, g_n\}$ and $\mathbf{g} = (g_1, \dots, g_n)^\top$. By the examples in section 3, we may assume that $n \geq 3$. Take an admissible array $\mathbf{g}||M$ with a nonsingular integer matrix $M = (a_{ij})$. The columns of M may be assumed to be as described in the preceding proof. The array

$(0, g_1)$	a_{11}	$\dots$	a_{1n}	-1	-1				
$(0, g_2)$	a_{21}		a_{2n}			-1			
$(0, g_3)$	a_{31}		a_{3n}				-1		
$\vdots$	$\vdots$		$\vdots$						
$(0, g_n)$	a_{n1}	$\dots$	a_{nn}						$\vdots$
$(1, 0)$				1	-1	-1	-1		
$(1, g_1)$				1	1				
$(1, g_2)$						1			
$(1, g_3)$							1		
$\vdots$									$\vdots$

is admissible for $\mathbf{Z}_2 \times G$. The matrix is upper block triangular with determinant $\det M \cdot 2 \neq 0$, and we have $d(\mathbf{Z}_2 \times G) = 2$. We turn to $\mathbf{Z}_3 \times G$. Suppose $g_1 + g_2 = 0$

and $g_1 + g_1 = g_3$. Then the array

$(0,g_1)$	a_{11}	$\dots$	a_{1n}			-1							
$(0,g_2)$	a_{21}		a_{2n}				-1			-1			
$(0,g_3)$	a_{31}		a_{3n}		-1			$\vdots$					
$\vdots$	$\vdots$		$\vdots$								-1		
$(0,g_n)$	a_{n1}	$\dots$	a_{nn}						-1				$\vdots$
$(1,0)$				1	0	-1	-1		-1				
$(1,g_1)$				1	1	1		$\vdots$					
$(2,g_1)$				-1	1	0				-1	-1		
$(1,g_2)$							1						
$\vdots$								$\vdots$					
$(1,g_n)$									1				$\vdots$
$(2,0)$										1			
$(2,g_2)$											1		
$\vdots$													$\vdots$

is admissible. The matrix of this array is upper block triangular. The determinant of the $n \times n$ block is nonzero, and the determinant of the 3×3 block equals -3. Thus, $\mathcal{L}(\mathbf{Z}_3 \times G)$ is well-rounded with $d(\mathbf{Z}_3 \times G) = 2$. We finally consider $\mathbf{Z}_4 \times G$. Let $g_1 + g_n = 0$. Now the array

$(0,g_1)$	a_{11}	$\dots$	a_{1n}			-1	-1	-1	-1				
$(0,g_2)$	a_{21}		a_{2n}							-1			
$\vdots$			$\vdots$										
$(0,g_n)$	a_{n1}	$\dots$	a_{nn}			-1							$\vdots$
$(1,0)$				1	-1	1	-1			-1			
$(2,0)$				1	1	0		-1					
$(3,0)$				-1	1	1			-1				
$(1,g_1)$							1						
$(2,g_1)$								1					
$(3,g_1)$									1				
$(1,g_2)$										1			
$\vdots$													$\vdots$

is admissible. The determinant of the 3×3 block is 4 and thus nonzero. It follows that $\mathcal{L}(\mathbf{Z}_4 \times G)$ is well-rounded with $d(\mathbf{Z}_4 \times G) = 2$. □

LEMMA 5.3. *The lattices $\mathcal{L}(\mathbf{Z}_2 \times \mathbf{Z}_4)$, $\mathcal{L}(\mathbf{Z}_3 \times \mathbf{Z}_3)$, and $\mathcal{L}(\mathbf{Z}_4 \times \mathbf{Z}_4)$ are well-rounded with minimum distance 2.*

Proof. The array

$$\begin{array}{c||cccccc|cc}
(1,0) & 1 & & & & 1 & & -1 \\
(1,3) & 1 & 1 & & & -1 & & \\
(0,3) & -1 & 1 & 1 & -1 & & -1 & \\
(1,2) & & -1 & 1 & 1 & -1 & & -1 \\
(1,1) & & & -1 & 1 & 1 & & \\
\hline
(0,1) & & & & & & 1 & 0 \\
(0,2) & & & & & & 1 & 1
\end{array}$$

is admissible for $\mathbf{Z}_2 \times \mathbf{Z}_4$, the determinants of the diagonal blocks being 8 and 1, which proves the assertion for $\mathbf{Z}_2 \times \mathbf{Z}_4$. The array

$$\begin{array}{c||cccccccc}
(0,1) & 1 & & & & & & -1 & 1 \\
(1,0) & 1 & 1 & & & & & & -1 \\
(1,1) & -1 & 1 & 1 & & & & & \\
(2,1) & & -1 & 1 & 1 & & & & \\
(0,2) & & & -1 & 1 & 1 & & & \\
(2,0) & & & & -1 & 1 & 1 & & \\
(2,2) & & & & & -1 & 1 & 1 & \\
(1,2) & & & & & & -1 & 1 & 1
\end{array}$$

is admissible for $\mathbf{Z}_3 \times \mathbf{Z}_3$, and since the determinant of the entire 8×8 matrix is -45, we get the assertion in this case. Finally, the array

$$\begin{array}{c||cccccc|cccccc|ccc}
(0,1) & 1 & & & & -1 & 1 & & & & & & & & & \\
(1,0) & 1 & 1 & & & & -1 & & & & & & & & & \\
(1,1) & -1 & 1 & 1 & & & & & & & & & & & & \\
(2,1) & & -1 & 1 & 1 & & & & & & & & & & & \\
(3,2) & & & -1 & 1 & 1 & & & & & & & & & & \\
(1,3) & & & & -1 & 1 & 1 & & & & & & & & & \\
\hline
(1,2) & & & & & & & 1 & & & & -1 & 1 & & & \\
(3,1) & & & & & & & 1 & 1 & & & & -1 & & & \\
(0,3) & & & & & & & -1 & 1 & 1 & & & & & & \\
(3,0) & & & & & & & & -1 & 1 & 1 & & & & & \\
(3,3) & & & & & & & & & -1 & 1 & 1 & & & & \\
(2,3) & & & & & & & & & & -1 & 1 & 1 & & & \\
\hline
(2,0) & & & & & & & & & & & & & 1 & -1 & 1 \\
(0,2) & & & & & & & & & & & & & 1 & 1 & -1 \\
(2,2) & & & & & & & & & & & & & -1 & 1 & 1
\end{array}$$

is admissible for $\mathbf{Z}_4 \times \mathbf{Z}_4$. The determinants of the diagonal blocks are -16, -16, 4. Consequently, $\mathcal{L}(\mathbf{Z}_4 \times \mathbf{Z}_4)$ is well-rounded with minimum distance 2. □

Now we can finish the game. Let

$$G = \underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{i} \times \underbrace{\mathbf{Z}_3 \times \cdots \times \mathbf{Z}_3}_{j} \times \underbrace{\mathbf{Z}_4 \times \cdots \times \mathbf{Z}_4}_{k} \times H,$$

where H contains only cyclic groups of order at least 5 or where H is absent. In the former case repeated application of Lemmas 5.1 and 5.2 shows that $\mathcal{L}(G)$ is well-rounded with $d(G) = 2$. We are left with the latter case. Since $\mathbf{Z}_2 \times \mathbf{Z}_3 = \mathbf{Z}_6$, $\mathbf{Z}_3 \times \mathbf{Z}_4 = \mathbf{Z}_{12}$, $\mathbf{Z}_2 \times \mathbf{Z}_4$ are well-rounded with minimum distance 2 (section 4 for the

first two and Lemma 5.3 for the last group), Lemmas 5.1 and 5.2 give the assertion if two of the numbers i, j, k are at least 1. It remains to consider the cases where G is one of the groups

$$G_2 = \underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{i}, \quad G_3 = \underbrace{\mathbf{Z}_3 \times \cdots \times \mathbf{Z}_3}_{j}, \quad G_4 = \underbrace{\mathbf{Z}_4 \times \cdots \times \mathbf{Z}_4}_{k}.$$

For $i = 1$ we are in Example 3.2, and for $i \geq 2$ we obtain from Example 3.4 and Lemma 5.2 that G_2 is as asserted. The case of G_3 is settled by Example 3.1 for $j = 1$ and by Lemmas 5.2 and 5.3 for $j \geq 2$. Example 3.3 ($k = 1$) and Lemmas 5.2 and 5.3 ($k \geq 2$) finally yield the assertion for G_4.

6. Direct products: Bases of minimal vectors. This section is devoted to the proof of Theorem 1.2. We want to emphasize once more that Theorem 1.2 was previously proved by Sha [20] for subgroups G of the direct product of two cyclic groups. In particular, Lemma 6.3 and results resembling Lemmas 6.1 and 6.2 in the cases of cyclic groups G and H were already established in [20] using arguments different from ours.

LEMMA 6.1. *Let G and H be finite Abelian groups such that $\mathcal{L}(G)$ and $\mathcal{L}(H)$ have bases of minimal vectors and such that $d(G) = d(H) = 2$. Also suppose that there are admissible arrays $\mathbf{g}||M_G$ and $\mathbf{h}||M_H$ coming from minimal basis vectors such that $\det M_G = \pm|G|$ and $\det M_H = \pm|H|$. Put $K = G \times H$. Then $\mathcal{L}(K)$ has a basis of minimal vectors, $d(K) = 2$, and there exists an admissible array $\mathbf{k}||M_K$ resulting from minimal basis vectors such that $\det M_K = \pm|K|$.*

Proof. Let $G, H, \mathbf{g}, \mathbf{h}$ be as in the proof of Lemma 5.1. Our present assumptions guarantee that the two matrices M_G and M_H in the proof of Lemma 5.1 may be taken so that $\widetilde{M}_G$ and $\widetilde{M}_H$ are basis matrices and so that $\det M_G = \pm(n+1)$ and $\det M_H = \pm(m+1)$.

Denote the matrix in the array in the proof of Lemma 5.1 by M_K. It is clear that $\det M_K = \pm|K|$. The extended matrices $\widetilde{M}_G$, $\widetilde{M}_H$, $\widetilde{M}_K$ are

$$\widetilde{M}_G = \begin{pmatrix} M_G \\ s \end{pmatrix}, \quad \widetilde{M}_H = \begin{pmatrix} M_H \\ t \end{pmatrix}, \quad \widetilde{M}_K = \begin{pmatrix} M_G & 0 & X \\ 0 & M_H & Y \\ 0 & 0 & I \\ s & t & e \end{pmatrix},$$

where $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_m)$ have entries from the set $\{-1, 0, 1\}$, $e = (1, \ldots, 1)$, I is the $nm \times nm$ identity matrix, and X, Y are the two blocks we also see in the array in the proof of Lemma 5.1. We have to show that $\widetilde{M}_K$ is a basis matrix for $\mathcal{L}(K)$, and since $|K| = (n+1)(m+1)$, this is equivalent to the equality $\det \widetilde{M}_K^\top \widetilde{M}_K = (n+1)^3(m+1)^3$.

We expand $\det \widetilde{M}_K^\top \widetilde{M}_K$ by the Cauchy–Binet formula. In what follows we also write $|A|$ for the determinant of a matrix A. We then have

$$|\widetilde{M}_K^\top \widetilde{M}_K| = |M_\ell|^2 + \sum_{j,k=1}^{n,m} |M_{jk}|^2 + \sum_k |M_{0,k}|^2 + \sum_j |M_{j,0}|^2,$$

the matrices on the right resulting from $\widetilde{M}_K$ after deleting the last row, the row labeled by (g_j, h_k), the row labeled by $(0, h_k)$, and the row labeled by $(g_j, 0)$, respectively. The matrix M_ℓ is upper block triangular and hence $|M_\ell|^2 = |M_G|^2|M_H|^2$. For $j, k \geq 1$, we

may expand the determinant $|M_{jk}|$ along the rows intersecting the identity matrix I, giving

$$|M_{jk}|^2 = \begin{vmatrix} M_G & 0 & X_j \\ 0 & M_H & Y_k \\ s & t & 1 \end{vmatrix}^2,$$

where X_j and Y_k are columns with a single -1 and zeros otherwise. Adding the first $n+m$ rows to the last row, we get

$$|M_{jk}|^2 = \begin{vmatrix} M_G & 0 & X_j \\ 0 & M_H & Y_k \\ 0 & 0 & -1 \end{vmatrix}^2 = |M_G|^2 |M_H|^2.$$

Expanding the determinant $|M_{0,k}|$ along the rows which intersect the identity matrix I we obtain

$$|M_{0,k}|^2 = \begin{vmatrix} M_G & 0 \\ p & B_{H,k} \end{vmatrix}^2, \quad p = \begin{pmatrix} 0 \\ s \end{pmatrix}, \quad B_{H,k} = \begin{pmatrix} M_{H,k} \\ t \end{pmatrix},$$

where $M_{H,k}$ arises from M_H by deleting the kth row. The matrix $B_{H,k}$ is square and hence

$$|M_{0,k}|^2 = |M_G|^2 |B_{H,k}|^2.$$

Analogously, $|M_{j,0}|^2 = |B_{G,j}|^2 |M_H|^2$. In summary,

$$|\widetilde{M}_K^\top \widetilde{M}_K| = |M_G|^2 |M_H|^2 + nm|M_G|^2|M_H|^2 + |M_G|^2 \sum_k |B_{H,k}|^2 + \sum_j |M_H|^2 |B_{G,j}|^2.$$

Again due to Cauchy–Binet,

$$|M_H|^2 + \sum_k |B_{H,k}|^2 = |\widetilde{M}_H^\top \widetilde{M}_H| = (m+1)^3,$$
$$|M_G|^2 + \sum_j |B_{G,j}|^2 = |\widetilde{M}_G^\top \widetilde{M}_G| = (n+1)^3,$$

and taking into account that $|M_G|^2 = (n+1)^2$ and $|M_H|^2 = (m+1)^2$, we arrive at the conclusion that $|\widetilde{M}_K^\top \widetilde{M}_K|$ is equal to

$$(n+1)^2(m+1)^2(1+nm) + (n+1)^2\Big((m+1)^3 - (m+1)^2\Big) + (m+1)^2\Big((n+1)^3 - (n+1)^2\Big),$$

which equals $(n+1)^3(m+1)^3$, as desired. □

In section 4 we showed that the hypothesis of Lemma 6.1 is satisfied if G and H are cyclic groups of order at least five. Successive application of Lemma 6.1 therefore gives Theorem 1.2 for all groups $\mathbf{Z}_{m_1} \times \cdots \times \mathbf{Z}_{m_k}$ with $m_1, \ldots, m_k \geq 5$.

LEMMA 6.2. *Let $m \in \{2,3,4\}$ and let G be a finite Abelian group such that $\mathcal{L}(G)$ has a basis of minimal vectors and such that $d(G) = 2$. Also suppose that there is an admissible array $\mathbf{g}||M_G$ coming from minimal basis vectors such that* $\det M_G = \pm|G|$. *Put $K = \mathbf{Z}_m \times G$. Then $\mathcal{L}(K)$ has a basis of minimal vectors, $d(K) = 2$, and there exists an admissible array $\mathbf{k}||M_K$ resulting from minimal basis vectors such that* $\det M_K = \pm|K|$.

Proof. We proceed as in the proof of the preceding lemma. Let G, $\mathbf{g}$, and the admissible arrays $\mathbf{k}||M_K$ be as in the proof of Lemma 5.2. These arrays are associated with vectors of minimum length 2 and the extended matrices $\widetilde{M}_K$ are of the form

$$\widetilde{M}_K = \begin{pmatrix} M_G & * & * \\ 0 & M_m & * \\ 0 & 0 & I \\ s & t & e \end{pmatrix}.$$

We already know that $\det M_G = \pm|G| = \pm(n+1)$ and $\det M_m = \pm m$. It remains to prove that $\det \widetilde{M}_K^\top \widetilde{M}_K = m^3(n+1)^3$.

We consider the case $m = 3$. The cases $m = 2$ and $m = 4$ may be disposed of in a completely analogous fashion. Expanding $\det \widetilde{M}_K^\top \widetilde{M}_K$ by the Cauchy–Binet formula we get

$$\det \widetilde{M}_K^\top \widetilde{M}_K = |M_\ell|^2 + \sum_{k=1}^{2n-1} |M_{I,k}|^2 + \sum_{j=1}^{3} |M_{3,j}|^2 + \sum_{i=1}^{n} |M_{0,i}|^2,$$

where M_ℓ results from deleting the last row, $M_{I,k}$ comes from deleting the row which contains the kth entry 1 of the $(2n-1)\times(2n-1)$ identity matrix I, $M_{3,j}$ arises from deleting the row containing the jth row of M_3, and $M_{0,i}$ emerges from deleting the ith row of M_G. Clearly, $|M_\ell|^2 = |M_G|^2|M_3|^2 = 9(n+1)^2$. Expanding $|M_{I,k}|$ along the rows intersecting the identity matrix and adding after that the first $n+3$ rows to the last row, we obtain

$$|M_{I,k}|^2 = \begin{vmatrix} M_G & * & * \\ 0 & M_3 & * \\ s & t & 1 \end{vmatrix}^2 = \begin{vmatrix} M_G & * & * \\ 0 & M_3 & * \\ 0 & 0 & -1 \end{vmatrix}^2 = |M_G|^2|M_3|^2 = 9(n+1)^2.$$

We expand $M_{3,j}$ again along the rows intersecting the identity matrix and then add the first $n+2$ rows to the last. It results that

$$|M_{3,j}|^2 = \begin{vmatrix} M_G & * \\ 0 & Q_{3,j} \\ s & t \end{vmatrix}^2 = \begin{vmatrix} M_G & * \\ 0 & Q_{3,j} \\ 0 & t_j \end{vmatrix}^2 = |M_G|^2 \begin{vmatrix} Q_{3,j} \\ t_j \end{vmatrix}^2 = (n+1)^2 \begin{vmatrix} Q_{3,j} \\ t_j \end{vmatrix}^2$$

with

$$\sum_{j=1}^{3} \begin{vmatrix} Q_{3,j} \\ t_j \end{vmatrix}^2 = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}^2 + \begin{vmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & -1 \end{vmatrix}^2 + \begin{vmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix}^2.$$

Each determinant on the right equals 3 and hence the sum of their squares is 27. Finally, again after expansion along the rows intersecting I and a row change,we get

$$|M_{0,i}|^2 = \begin{vmatrix} B_{G,i} & * \\ 0 & M_3 \end{vmatrix}^2 = |B_{G,i}|^2|M_3|^2 = 9|B_{G,i}|^2,$$

where $B_{G,i}$ is the square matrix obtained from M_G by deleting the ith row. By the Cauchy–Binet formula,

$$\sum_{j=1}^{n} |B_{G,i}|^2 = |\widetilde{M}_G^\top \widetilde{M}_G| - |M_G|^2 = (n+1)^3 - (n+1)^2 = (n+1)^2 n.$$

Putting things together we see that

$$\det \widetilde{M}_K^\top \widetilde{M}_K = 9(n+1)^2(1+2n-1) + 27(n+1)^2 + 9(n+1)^2 n = 3^3(n+1)^3,$$

which is what we wanted. ∎

Lemma 6.3. *Let G be one of the groups $\mathbf{Z}_2 \times \mathbf{Z}_4$, $\mathbf{Z}_3 \times \mathbf{Z}_3$, $\mathbf{Z}_4 \times \mathbf{Z}_4$. Then $\mathcal{L}(G)$ has a basis of minimal vectors, $d(G) = 2$, and there exists an admissible array $\mathbf{g}||M_G$ coming from minimal basis vectors such that $\det M = \pm|G|$ and $\det \widetilde{M}_G^\top \widetilde{M}_G = |G|^3$.*

Proof. The admissible array $\mathbf{g}||M_7$ we have shown for $G = \mathbf{Z}_2 \times \mathbf{Z}_4$ in the proof of Lemma 5.3 satisfies $\det M_7 = 8$ and $\det \widetilde{M}_7^\top \widetilde{M}_7 = 8^3$. The array

$$\mathbf{g}||M_8 = \begin{array}{c||cccc|cccc}
(0,1) & 1 & & & & -1 & & -1 & \\
(0,2) & & 1 & 1 & & & -1 & & \\
(1,0) & 1 & & & 1 & 1 & 1 & & \\
(1,1) & -1 & & & 1 & & & & \\
\hline
(1,2) & & & & & & & 1 & -1 \\
(2,0) & & 1 & -1 & & & & & 1 \\
(2,1) & & & 1 & -1 & 1 & & & \\
(2,2) & & -1 & & & & 1 & 1 & 1
\end{array}$$

is admissible for $\mathbf{Z}_3 \times \mathbf{Z}_3$, and we have $\det M_8 = 9$ and $\det \widetilde{M}_8^\top \widetilde{M}_8 = 9^3$. The array $\mathbf{g}||M_{15}$ given by

$$\begin{array}{c|ccc|cccc|cccc|cccc}
(0,1) & 1 & 1 & 1 & 1 & & & & & & & & & & & \\
(0,2) & & & & & 1 & & & & & & & & & & \\
(0,3) & & & & & & 1 & 1 & 1 & & & & & & & \\
\hline
(1,0) & & -1 & & & 1 & -1 & & & & & & & & & \\
(1,1) & & & & & & 1 & & & & & & & & & \\
(1,2) & 1 & & & & -1 & & & & & & & -1 & & -1 & \\
(1,3) & -1 & 1 & & & & & & & 1 & 1 & 1 & & -1 & & \\
\hline
(2,0) & & & & -1 & & & & & & & & 1 & 1 & & \\
(2,1) & & & & & & & & & 1 & & & & & 1 & \\
(2,2) & & & 1 & & & & & & & 1 & & & & & \\
(2,3) & & & -1 & 1 & & & & & & & 1 & & & & -1 \\
\hline
(3,0) & & & & & & & -1 & & -1 & & & & & & \\
(3,1) & & & & & & & 1 & -1 & & -1 & & & & 1 & 1 \\
(3,2) & & & & & & & & 1 & & & -1 & 1 & & & 1 \\
(3,3) & & & & & & & & & & & & & 1 & &
\end{array}$$

is admissible for $\mathbf{Z}_4 \times \mathbf{Z}_4$ with $\det M_{15} = -16$ and $\det \widetilde{M}_{15}^\top \widetilde{M}_{15} = 16^3$. ∎

The proof of Theorem 1.2 may now be completed as at the end of section 5.

7. Bounds for the covering radius. In this section we study the lattices $\mathcal{L}_n := \mathcal{L}(\mathbf{Z}_m)$ $(m = n+1)$ with the goal of proving Theorem 1.3. For $n \geq 2$, let $B_{n+1,n}$ be the $(n+1) \times n$ version of the matrices

$$B_{3,2} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \\ 1 & 1 \end{pmatrix}, \; B_{4,3} = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{pmatrix}, \; B_{5,4} = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Note that in section 4 we denoted these matrices by $\widetilde{T}_{n+1}$. In other words, we now denote $\widetilde{T}_m$ by $B_{m,m-1}$. The $(n+1)\times k$ matrix formed by the first k columns of $B_{n+1,n}$ is denoted by $B_{n+1,k}$.

Example 7.1. Let us begin with an example. Consider $G = \mathbf{Z}_4$. We know from section 4 that

$$B_{4,3}\ (=\widetilde{T}_4)\ = \left(\begin{array}{ccc} -2 & 1 & 0 \\ 1 & -2 & 1 \\ \hline 0 & 1 & -2 \\ \hline 1 & 0 & 1 \end{array}\right)$$

is a basis matrix for the lattice $\mathcal{L}_3 := \mathcal{L}(G)$. This follows from the fact that

$$V_3 := \sqrt{\det B_{4,3}^\top B_{4,3}} = 8 = 4^{3/2}.$$

Let $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ be the columns of $B_{4,3}$. Then

$$B_{4,2} = (\mathbf{b}_1\ \mathbf{b}_2) = \left(\begin{array}{cc} -2 & 1 \\ 1 & -2 \\ \hline 0 & 1 \\ \hline 1 & 0 \end{array}\right), \quad B_{4,1} = (\mathbf{b}_1) = \left(\begin{array}{c} -2 \\ 1 \\ \hline 0 \\ \hline 1 \end{array}\right).$$

Let further $\mathcal{L}_2$ and $\mathcal{L}_1$ be the sublattices of $\mathcal{L}_3$ spanned by the columns of $B_{4,2}$ and $B_{4,1}$. The determinants of $\mathcal{L}_2$ and $\mathcal{L}_1$ are

$$V_2 = \sqrt{\det B_{4,2}^\top B_{4,2}} = \left|\begin{array}{cc} 6 & -4 \\ -4 & 6 \end{array}\right|^{1/2} = \sqrt{20}, \quad V_1 = \sqrt{\det B_{4,1}^\top B_{4,1}} = \sqrt{6}.$$

The lattice $\mathcal{L}_1$ is spanned by a vector of length $\sqrt{6}$ and can therefore be covered by 1-dimensional balls of radius $r_1 = \sqrt{6}/2$ centered at the lattice points. Now consider an arbitrary point x in $\mathrm{span}_{\mathbf{R}}\mathcal{L}_2$. We may assume that this point lies between the two lines $\mathrm{span}_{\mathbf{R}}\mathcal{L}_1$ and $\mathbf{b}_2 + \mathrm{span}_{\mathbf{R}}\mathcal{L}_1$. Let h_1 be the distance between these two lines. The distance between x and one of the two lines is at most $h_1/2$. This implies that x is contained in a 2-dimensional ball of radius $r_2 \le \sqrt{r_1^2 + (h_1/2)^2}$ centered at a lattice point of $\mathcal{L}_1$ or $\mathbf{b}_2 + \mathcal{L}_1$. Since the area of a parallelogram is the product of the length of the baseline and the height, we have $V_2 = V_1 h_1$. Thus, $\mathrm{span}_{\mathbf{R}}\mathcal{L}_2$ may be covered by 2-dimensional balls of radius r_2 centered at the points of $\mathcal{L}_2$, where

$$r_2^2 \le r_1^2 + \left(\frac{V_2}{2V_1}\right)^2 = \frac{6}{4} + \frac{20}{4\cdot 6} = \frac{7}{3}.$$

Now take a point y in $\mathrm{span}_{\mathbf{R}}\mathcal{L}_3$, without loss of generality, between the two planes $\mathrm{span}_{\mathbf{R}}\mathcal{L}_2$ and $\mathbf{b}_3 + \mathrm{span}_{\mathbf{R}}\mathcal{L}_2$. Letting h_2 denote the distance between these two planes, there is a point in $\mathcal{L}_2$ or $\mathbf{b}_3 + \mathcal{L}_2$ whose distance to y is at most $\sqrt{r_2^2 + (h_2/2)^2}$. Since $V_3 = V_2 h_2$, we conclude that $\mathrm{span}_{\mathbf{R}}\mathcal{L}_3$ may be covered by 3-dimensional balls of radius r_3 with the centers at the points of $\mathcal{L}_3$, where

$$r_3^2 \le r_2^2 + \left(\frac{V_3}{2V_2}\right)^2 \le \frac{7}{3} + \frac{64}{4\cdot 20} = \frac{47}{15}.$$

Consequently, $\mu(\mathbf{Z}_4) \le \sqrt{47/15} \approx 1.7701$.

PROPOSITION 7.2. *Let $\mathbf{b}_1, \ldots, \mathbf{b}_n$ be points in the root lattice $\mathcal{A}_n$ such that*

$$\mathrm{span}_{\mathbf{R}}\{\mathbf{b}_1, \ldots, \mathbf{b}_n\} = \mathrm{span}_{\mathbf{R}}\mathcal{A}_n.$$

For $k = 1, \dots, n$, denote by $\mathcal{L}_k$ the lattice spanned by $\mathbf{b}_1, \dots, \mathbf{b}_k$, let $C_{n+1,k}$ stand for the $(n+1) \times k$ matrix whose columns are the coordinates of $\mathbf{b}_1, \dots, \mathbf{b}_k$, and put

$$V_k = \sqrt{\det C_{n+1,k}^{\top} C_{n+1,k}}.$$

If $\operatorname{span}_{\mathbf{R}} \mathcal{L}_k$ $(1 \le k \le n-1)$ can be covered by k-dimensional balls of radius r_k centered at the points of $\mathcal{L}_k$, then $\operatorname{span}_{\mathbf{R}} \mathcal{L}_{k+1}$ can be covered by balls of dimension $k+1$ centered at the points of $\mathcal{L}_{k+1}$ whose radius r_{k+1} satisfies

$$r_{k+1}^2 \le r_k^2 + \left(\frac{V_{k+1}}{2V_k} \right)^2,$$

and consequently,

$$r_n^2 \le r_1^2 + \left(\frac{V_2}{2V_1} \right)^2 + \cdots + \left(\frac{V_n}{2V_{n-1}} \right)^2.$$

Proof. This can be shown by the argument employed in Example 7.1. □

The only problem in general is the computation of the determinants V_k. Fortunately, this is easy for $\mathcal{L}_n = \mathcal{L}(\mathbf{Z}_m)$ $(m = n+1)$, in which case the matrices $C_{n+1,k}$ are just the matrices $B_{n+1,k}$ we introduced above. We also need the $n \times n$ versions Q_n of the matrices

$$Q_2 = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}, \; Q_3 = \begin{pmatrix} 6 & -4 & 2 \\ -4 & 6 & -4 \\ 2 & -4 & 6 \end{pmatrix}, \; Q_4 = \begin{pmatrix} 6 & -4 & 1 & 1 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 1 & 1 & -4 & 6 \end{pmatrix},$$

$$Q_5 = \begin{pmatrix} 6 & -4 & 1 & 0 & 1 \\ -4 & 6 & -4 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \\ 0 & 1 & -4 & 6 & -4 \\ 1 & 0 & 1 & -4 & 6 \end{pmatrix}, \; Q_6 = \begin{pmatrix} 6 & -4 & 1 & 0 & 0 & 1 \\ -4 & 6 & -4 & 1 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 1 & -4 & 6 & -4 \\ 1 & 0 & 0 & 1 & -4 & 6 \end{pmatrix}.$$

Finally, for $k \ge 1$, we denote by R_k the $k \times k$ version of the matrices $R_1 = (6)$,

$$R_2 = \begin{pmatrix} 6 & -4 \\ -4 & 6 \end{pmatrix}, \; R_3 = \begin{pmatrix} 6 & -4 & 1 \\ -4 & 6 & -4 \\ 1 & -4 & 6 \end{pmatrix}, \; R_4 = \begin{pmatrix} 6 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 6 \end{pmatrix},$$

$$R_5 = \begin{pmatrix} 6 & -4 & 1 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \\ 0 & 1 & -4 & 6 & -4 \\ 0 & 0 & 1 & -4 & 6 \end{pmatrix}, \; R_6 = \begin{pmatrix} 6 & -4 & 1 & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 1 & -4 & 6 & -4 \\ 0 & 0 & 0 & 1 & -4 & 6 \end{pmatrix}.$$

LEMMA 7.3. *For $n \ge 2$ and $1 \le k \le n-1$,*

$$B_{n+1,n}^{\top} B_{n+1,n} = Q_n, \quad \det Q_n = (n+1)^3,$$

$$B_{n+1,k}^{\top} B_{n+1,k} = R_k, \quad \det R_k = \frac{(k+1)(k+2)^2(k+3)}{12}.$$

Proof. The formulas for the products of the matrices can be verified by straightforward computation. The formula for $\det Q_n$ is nothing but Lemma 4.2. The formula for $\det R_k$ was first established in [5]. Proofs of that formula can also be found in [6, Theorem 10.59] and [7]. □

Proof of Theorem 1.3. We know from section 4 that $B_{n+1,n} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ is a basis matrix for $\mathcal{L}_n := \mathcal{L}(\mathbf{Z}_{n+1})$. Let $\mathcal{L}_k$ be the sublattices as in Proposition 7.2. The 1-dimensional lattice $\mathcal{L}_1$ is spanned by a vector of length $\sqrt{6}$. We may therefore use Proposition 7.2 with $r_1 = \sqrt{6}/2$ to obtain that

$$\mu(\mathcal{L}_n)^2 \le \frac{6}{4} + \left(\frac{V_2}{2V_1}\right)^2 + \cdots + \left(\frac{V_n}{2V_{n-1}}\right)^2 = \frac{6}{4} + \frac{1}{4}\sum_{k=1}^{n-2}\frac{V_{k+1}^2}{V_k^2} + \frac{1}{4}\,\frac{V_n^2}{V_{n-1}^2}.$$

From Lemma 7.3 we see that if $1 \le k \le n-2$, then

$$\begin{aligned}\frac{V_{k+1}^2}{V_k^2} &= \frac{\det B_{n+1,k+1}^\top B_{n+1,k+1}}{\det B_{n+1,k}^\top B_{n+1,k}} = \frac{(k+2)(k+3)^2(k+4)}{(k+1)(k+2)^2(k+3)} \\ &= \frac{(k+3)(k+4)}{(k+1)(k+2)} = 1 + \frac{2(2k+5)}{(k+1)(k+2)} = 1 + 2\left(\frac{3}{k+1} - \frac{1}{k+2}\right).\end{aligned}$$

Consequently,

$$\begin{aligned}\sum_{k=1}^{n-2}\frac{V_{k+1}^2}{V_k^2} &= n - 2 + 2\sum_{k=1}^{n-2}\left(\frac{3}{k+1} - \frac{1}{k+2}\right) = n - 2 + 2\left(\frac{3}{2} + 2\sum_{k=3}^{n-1}\frac{1}{k} - \frac{1}{n}\right) \\ &= n + 1 - \frac{2}{n} + 4\sum_{k=3}^{n-1}\frac{1}{k} < n + 1 - \frac{2}{n} + 4\int_2^{n-1}\frac{dx}{x} \\ &= n + 1 - \frac{2}{n} + 4\log(n-1) - 4\log 2.\end{aligned}$$

Lemma 7.3 also implies that

$$\frac{V_n^2}{V_{n-1}^2} = \frac{\det B_{n+1,n}^\top B_{n+1,n}}{\det B_{n+1,n-1}^\top B_{n+1,n-1}} = \frac{12(n+1)^3}{n(n+1)^2(n+2)} = \frac{12(n+1)}{n(n+2)}.$$

In summary,

$$\begin{aligned}\mu(\mathcal{L}_n)^2 &< \frac{1}{4}\left(6 + n + 1 - \frac{2}{n} + 4\log(n-1) - 4\log 2 + \frac{12(n+1)}{n(n+2)}\right) \\ &= \frac{1}{4}\left(n + 4\log(n-1) + 7 - 4\log 2 + \frac{10n+8}{n(n+2)}\right),\end{aligned}$$

and since $(10n+8)/(n(n+2)) < 10/n$, we arrive at the asserted bound. □

We remark that Example 7.1 gives $\mu(\mathbf{Z}_4) < 1.7701$ whereas the table presented in section 1 shows the slightly worse bound $\mu(\mathbf{Z}_4) < 1.8257$. This discrepancy is caused by the circumstance that in Example 7.1 we didn't estimate a sum by an integral.

As already mentioned in the introduction, Sha [20] showed that $\mu(G) \le \mu(\mathcal{A}_n) + \sqrt{2}$ if G is a group coming from elliptic curves over finite fields. Actually, his proof works for arbitrary finite Abelian groups G. It goes as follows. Let $\xi = (\xi_1, \ldots, \xi_n, \xi_0) \in \mathrm{span}_{\mathbf{R}}\mathcal{A}_n$, where $\xi_0 := -\xi_1 - \cdots - \xi_n$, and pick $v = (v_1, \ldots, v_n, v_0) \in \mathcal{A}_n$ as a point for which $\|\xi - v\| \le \mu(\mathcal{A}_n)$. Then one may proceed as in [10, proof of

Theorem 3.4]. Namely, let $v_1g_1 + \cdots + v_ng_n = g_j$. If g_j is not the zero of the group, put

$$x = (x_1, \ldots, x_n, x_0) = (v_1, \ldots, v_{j-1}, v_j - 1, v_{j+1}, \ldots, v_n, v_0 + 1).$$

Then $x \in \mathcal{L}(G)$ and $\|v - x\| = \sqrt{2}$. In the case g_j is the zero of the group, let $x = v$, so that $x \in \mathcal{L}(G)$ and $\|v - x\| = 0$. In either case, $\|\xi - x\| \leq \mu(\mathcal{A}_n) + \sqrt{2}$.

The only difference between the arguments in [10] and [20] is that in [10] the point $v = (v_1, \ldots, v_n, v_0) \in \mathcal{A}_n$ was chosen so that v_i is the nearest integer to ξ_i for $i = 1, \ldots, n$. To ensure that v is in $\mathcal{A}_n$, one had to take $v_0 = -v_1 - \cdots - v_n$, and as the difference between ξ_0 and v_0 may be large, the bound for the covering radius obtained in [10] was too coarse. Sha's clever choice of $v = (v_1, \ldots, v_n, v_0) \in \mathcal{A}_n$ as a point for which $\|\xi - v\| \leq \mu(\mathcal{A}_n)$ remedied this defect.

8. The automorphism group. In this section we start out with the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $G = \{0, g_1, \ldots, g_n\}$ be a finite Abelian group and recall that

$$\mathcal{L}(G) = \left\{ X = \left(x_1, \ldots, x_n, -\sum_{i=1}^{n} x_i \right) \in \mathbf{Z}^{n+1} : \sum_{i=1}^{n} x_i g_i = 0 \right\}.$$

Every automorphism of G fixes 0 and permutes the elements $g_1, \ldots, g_n$. Hence $\mathrm{Aut}(G)$ can be identified (via a canonical isomorphism) with a subgroup of the symmetric group S_n. We denote this subgroup by H. Our objective is to construct a group isomorphism $\Phi : H \to \mathrm{Aut}(\mathcal{L}(G)) \cap S_n$, where $\mathrm{Aut}(\mathcal{L}(G))$ on the right is identified with a subgroup of $\mathrm{GL}_n(\mathbf{Z})$ as described in section 1 and S_n on the right is viewed in the natural fashion as the subgroup of the permutation matrices in $\mathrm{GL}_n(\mathbf{Z})$.

Let $\sigma \in H$. Then, for every $g_i \in G$, $\sigma(g_i) = g_{\sigma(i)}$ and $\sigma(0) = 0$. If

$$X = \left(x_1, \ldots, x_n, -\sum_{i=1}^{n} x_i \right) \in \mathcal{L}(G),$$

then $\sum_{i=1}^{n} x_i g_i = 0$. Notice that σ^{-1} is also in H, and so

$$0 = \sigma^{-1}(0) = \sum_{i=1}^{n} x_i g_{\sigma^{-1}(i)} = \sum_{i=1}^{n} x_{\sigma(i)} g_i.$$

Now define $\tau = \Phi(\sigma)$ on $\mathcal{L}(G)$ by

$$\tau \left(x_1, \ldots, x_n, -\sum_{i=1}^{n} x_i \right) := \left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, -\sum_{i=1}^{n} x_{\sigma(i)} \right).$$

It is clear that τ maps $\mathcal{L}(G)$ onto itself. The matrix $U \in \mathrm{GL}_n(\mathbf{Z})$ corresponding to τ as described in section 1 is obviously a permutation matrix. Consequently, τ is in $\mathrm{Aut}(\mathcal{L}(G)) \cap S_n$. Finally, it is readily seen that Φ is an injective group homomorphism. Hence $\Phi(H) \leq \mathrm{Aut}(\mathcal{L}(G)) \cap S_n$.

It remains to show that $\Phi(H) = \mathrm{Aut}(\mathcal{L}(G)) \cap S_n$. So suppose $\tau \in \mathrm{Aut}(\mathcal{L}(G)) \cap S_n$. If

$$X = \left(x_1, \ldots, x_n, -\sum_{i=1}^{n} x_i \right) \in \mathcal{L}(G),$$

then $\tau(X) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}, -\sum_{i=1}^{n} x_{\sigma(i)})$ with some $\sigma \in S_n$, and since both X and $\tau(X)$ belong to $\mathcal{L}(G)$, it follows that

$$0 = \sum_{i=1}^{n} x_i g_i = \sum_{i=1}^{n} x_{\sigma(i)} g_i.$$

We have $\tau = \Phi(\sigma)$ with $\sigma : G \to G$ defined by $\sigma(g_i) := g_{\sigma(i)}$ and $\sigma(0) := 0$. To complete the proof, we only need to show that σ is a group homomorphism, i.e., that

$$\sigma(g_i + g_j) = g_{\sigma(i)} + g_{\sigma(j)}.$$

Since $g_i + g_j \in G$, there must be some $g_k \in G$ such that $g_i + g_j = g_k$. In other words

$$g_i + g_j - g_k = 0.$$

Therefore the vector X with ith and jth coordinates equal to 1, kth coordinate equal to -1, $(n+1)$st coordinate equal to $-(1+1-1) = -1$, and the rest of the coordinates equal to 0, must be in $\mathcal{L}(G)$. Hence the vector $\tau(X)$ also lies in $\mathcal{L}(G)$. This vector has $\sigma(i)$th and $\sigma(j)$th coordinates equal to 1, $\sigma(k)$th coordinate equal to -1, $(n+1)$st coordinate equal to -1, and the rest of the coordinates equal to 0. This means that the equality

$$g_{\sigma(i)} + g_{\sigma(j)} - g_{\sigma(k)} = 0$$

must be satisfied in G and, hence,

$$\sigma(g_i + g_j) = \sigma(g_k) = g_{\sigma(k)} = g_{\sigma(i)} + g_{\sigma(j)}.$$

In summary, $\sigma \in H$, and so $\mathrm{Aut}(\mathcal{L}(G)) \cap S_n = \Phi(H)$, as desired. □

Theorem 1.4 has an interesting geometric interpretation in terms of the theory of quadratic forms (see, for instance, [21] for a detailed account of this subject and its connections to lattice theory). A real quadratic form in n variables $X = (x_1, \ldots, x_n)^\top$ can always be written in a unique way as

$$q(X) = X^\top A X,$$

where A is an $n \times n$ real symmetric matrix. Hence the space of real quadratic forms in n variables can be identified with the space of their coefficient matrices, which is the $\binom{n+1}{2}$-dimensional real vector space $\mathcal{S}^n$ of $n \times n$ real symmetric matrices. The set of positive definite forms $\mathcal{S}^n_{>0}$ is an open convex cone in $\mathcal{S}^n$ given by n polynomial inequalities (the Sylvester criterion).

Let B be an $m \times n$ real matrix of rank n, $1 \le n \le m$, then $\mathcal{L} = B\mathbf{Z}^n$ is a lattice of rank n in $\mathbb{R}^m$. The so-called norm form of $\mathcal{L}$, corresponding to the choice of the basis matrix B, is defined as the positive definite quadratic form in n variables, given by

$$q_B(X) = X^\top (B^\top B) X.$$

The function $B \mapsto B^\top B$ induces a bijection between the space of lattices (up to isometry) and the cone $\mathcal{S}^n_{>0}$ of positive definite quadratic forms (up to arithmetic equivalence).

Given a form $q \in \mathcal{S}^n$, its automorphism group is defined by

$$\mathrm{Aut}(q) := \{\tau \in \mathrm{GL}_n(\mathbf{Z}) : q(\tau(X)) = q(X) \text{ for all } X \in \mathbb{R}^n\}.$$

This is a finite group: it is contained in the intersection of the discrete group $\mathrm{GL}_n(\mathbf{Z})$ with the compact group $\mathcal{O}_n(\mathbb{R})$ of real orthogonal matrices. If $q \in \mathcal{S}^n_{>0}$ and $\mathcal{L}$ is the corresponding lattice, then $\mathrm{Aut}(q) = \mathrm{Aut}(\mathcal{L})$, the automorphism group of $\mathcal{L}$. Furthermore, given any finite subgroup H of $\mathrm{GL}_n(\mathbf{Z})$, there exists a $q \in \mathcal{S}^n_{>0}$ (and hence a lattice) with $H \leq \mathrm{Aut}(q)$. Indeed, if $H \leq \mathrm{GL}_n(\mathbf{Z})$ is a finite group and $f \in \mathcal{S}^n_{>0}$, then the form defined by

$$q(X) := \sum_{\tau \in H} f(\tau(X))$$

is in $\mathcal{S}^n_{>0}$ and $H \leq \mathrm{Aut}(q)$. Finally, for a fixed finite subgroup H of $\mathrm{GL}_n(\mathbf{Z})$, define

$$\mathcal{B}(H) = \{q \in \mathcal{S}^n : H \leq \mathrm{Aut}(q)\}.$$

The set $\mathcal{B}(H)$ is not empty by the above remark, and hence it is easily seen to be a real vector space. It is called the Bravais manifold of H. Define also the open convex polyhedral cone $\mathcal{B}_{>0}(H) = \mathcal{B}(H) \cap \mathcal{S}^n_{>0}$, which can be identified with the set of all lattices whose automorphism groups contain H.

Investigation of properties of Bravais manifolds corresponding to different finite subgroups of $\mathrm{GL}_n(\mathbf{Z})$ is of interest in lattice theory. Our Theorem 1.4 implies that the lattice $\mathcal{L}(G)$ coming from an Abelian group G of order $n+1$ via our construction is contained in the Bravais cone $\mathcal{B}_{>0}(\mathrm{Aut}(G))$.

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