SIMULTANEOUS DIOPHANTINE APPROXIMATION ON A RATIONAL ELLIPSE

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Abstract. We discuss simultaneous Diophantine approximation of points on the unit circle by rational points on the same circle, and some of its consequences. This text is included in [1].

Hlawka [2] proved the following theorem on simultaneous Diophantine approximation by quotients of Pythagorean triples.

Theorem 1. *Let* $x \in (0,1)$ *be a real number. Then there exist infinitely many Pythagorean triples* $\left(p, \sqrt{q^2-p^2}, q\right)$ *such that*

$$\left| x - \frac{p}{q} \right| \leq \frac{2\sqrt{2}}{q}. \tag{1}$$

Remark 1. In fact, the constant in the upper bound of Theorem 1 can be improved at least when $x = \frac{\sqrt{3}}{2}$. In [1] (Lemma 5.1) I proved that there exist infinitely many Pythagorean triples $\left(p, \sqrt{q^2-p^2}, q\right)$ such that

$$\frac{1}{(2+\sqrt{3})q} - \frac{2}{(2+\sqrt{3})q^2} < \left| \frac{\sqrt{3}}{2} - \frac{p}{q} \right| < \frac{1}{2\sqrt{3}q}, \tag{2}$$

explicitly constructing such an infinite sequence of Pythagorean triples.

We can use Theorem 1 to approximate points on a unit circle with rational points on the same circle.

Corollary 2. *Let* (x,y) *be a point on the unit circle. Then either* $x, y \in \{0, \pm 1\}$, *or there exist infinitely many rational points* $(p/q, r/q)$ *on the same circle such that*

$$\max\left\{ \left| x - \frac{p}{q} \right|, \left| y - \frac{r}{q} \right| \right\} \leq \frac{2\sqrt{2}}{q}. \tag{3}$$

Proof. First notice that it suffices to prove the statement of this corollary for the case $0 < x, y < 1$, namely the case when the point in question lies in the first quadrant, since any other point on the circle can be obtained from those in the first quadrant by a rational rotation. Let c be an arbitrary real number in the interval $(0,1)$, then either

$$0 < x \leq \sqrt{1-c^2} < 1,\ c \leq y < 1, \tag{4}$$

or

$$0 < y \leq \sqrt{1-c^2} < 1,\ c \leq x < 1. \tag{5}$$

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First assume that (4) holds. By Theorem 1, there exist infinitely many Pythagorean triples (p, r, q) with $r = \sqrt{q^2 - p^2}$ which satisfy (1). Then:

$$\frac{2\sqrt{2}}{q} \geq \left| x - \frac{p}{q} \right| = \left| \sqrt{1-y^2} - \sqrt{1 - \frac{r^2}{q^2}} \right| = \frac{\left| \frac{r^2}{q^2} - y^2 \right|}{\sqrt{1-y^2} + \sqrt{1-\frac{r^2}{q^2}}}$$

$$= \frac{\frac{r}{q} + y}{\sqrt{1-y^2} + \sqrt{1-\frac{r^2}{q^2}}} \left| y - \frac{r}{q} \right| \geq \frac{c\left(1 + \frac{n}{n+1}\right)}{2\sqrt{1 - \frac{n^2}{(n+1)^2}c^2}} \left| y - \frac{r}{q} \right|. \tag{6}$$

The last inequality is true because $\frac{w+z}{\sqrt{1-w^2}+\sqrt{1-z^2}}$ is an increasing function in both variables for $0 < z, w < 1$; since $y \geq c$, we can pick q large enough so that r/q would have to be sufficiently close to y so that $r/q \geq \frac{n}{n+1}c$ for some $n \in \mathbb{Z}_{>0}$, then $r/q + y \geq c\left(1 + \frac{n}{n+1}\right)$, and $\sqrt{1-y^2} + \sqrt{1 - \frac{r^2}{q^2}} \leq 2\sqrt{1 - \frac{n^2}{(n+1)^2}c^2}$. Then (6) implies:

$$\left| y - \frac{r}{q} \right| \leq \frac{\sqrt{1 - \frac{n^2}{(n+1)^2}c^2}}{c\left(1 + \frac{n}{n+1}\right)} \times \frac{4\sqrt{2}}{q}. \tag{7}$$

Since our choice of $c \in (0, 1)$ and positive integer n was arbitrary, we can for instance choose

$$c = \frac{2n+2}{\sqrt{8n^2+4n+1}}, \tag{8}$$

and take $n = 2$, in which case, combining (1), (7), and (8), we obtain (3).

If, on the other hand, (5) holds instead of (4), simply repeat the above argument interchanging x with y and p/q with r/q. This completes the proof. □

A related result has also been obtained by Kopetzky in [3] (also see [4]), however his bounds are different in flavor in the sense that the constants in the upper bounds depend on x and y. Notice that the bound of Corollary 2 can be easily extended to any rational ellipse.

Corollary 3. *Let (x, y) be a point on the ellipse E, given by the equation*

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1,$$

where a, b are positive rational numbers. Then either $(x, y) = (\pm a, 0), (0, \pm b)$, or there exist infinitely many rational points $(p/q, r/q)$ on the same ellipse such that

$$\max\left\{ \left| x - \frac{p}{q} \right|, \left| y - \frac{r}{q} \right| \right\} \leq \frac{2\sqrt{2}\max\{a, b\}}{q}. \tag{9}$$

Proof. Notice that the map $(x, y) \mapsto (x/a, y/b)$ is a bijection between E and the unit circle, which takes rational points to rational points. Now apply Corollary 2 to points of the form $(x/a, y/b)$. □

References

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