COVERING POINT-SETS WITH PARALLEL HYPERPLANES AND SPARSE SIGNAL RECOVERY

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Abstract. We give a new deterministic construction of integer sensing matrices that can be used for the recovery of integer-valued signals in compressed sensing. This is a family of $n \times d$ integer matrices, $d \geqslant n$, with bounded sup-norm and the property that no ℓ column vectors are linearly dependent, $\ell \leqslant n$. Further, if $\ell \leqslant o(\log n)$ then $d/n \to \infty$ as $n \to \infty$. Our construction comes from particular sets of difference vectors of point-sets in $\mathbb{R}^n$ that cannot be covered by few parallel hyperplanes. We construct examples of such sets on the $0, \pm 1$ grid and use them for the matrix construction. We also show a connection of our constructions to a simple version of the Tarski's plank problem.

1. Introduction and main results

An $n \times d$ real matrix A is said to be a sensing matrix for ℓ-sparse signals, $1 \leqslant \ell \leqslant n$, if for every nonzero vector $\boldsymbol{x} \in \mathbb{R}^d$ with no more than ℓ nonzero coordinates, $A\boldsymbol{x} \neq \mathbf{0}$. This is equivalent to saying that no ℓ columns of A are linearly dependent: such matrices $A = (a_{ij})$ are extensively used in the area of compressive sensing (see, for instance [5]), where the goal is to have d as large as possible with respect to n while (in the case A is an integer matrix),

$$|A| := \max |a_{ij}|$$

is small. Indeed, if we have such a matrix A and two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ with no more than $\ell/2$ nonzero coordinates each, then it is easy to see that $A\boldsymbol{x} = A\boldsymbol{y}$ if and only if $\boldsymbol{x} = \boldsymbol{y}$. Integer $n \times d$ matrices A with $d > n$ and all nonzero minors were recently studied in [6], [7], [8], [12] in the context of integer sparse recovery. The advantage of using integer matrices and integer signals is that in this situation if $A\boldsymbol{x} \neq \mathbf{0}$ then $\|A\boldsymbol{x}\| \geqslant 1$, which allows for robust error correction. In this paper, we prove the following theorem.

Theorem 1.1. *For all sufficiently large n, there exist $n \times d$ integer sensing matrices A for ℓ-sparse vectors, $1 \leqslant \ell \leqslant n-1$, such that $|A| = 2$ and*

$$d \geqslant \left(\frac{n+2}{2}\right)^{1+\frac{2}{3\ell-2}}.$$

An important implication of Theorem 1.1 is that when $\ell = o(\log n)$, then $d/n \to \infty$ as $n \to \infty$, meaning that d is super-linear in n. It is interesting to compare this observation to the previous results on integer sensing matrices obtained in [6], [7], [8]. While the matrices obtained there work for n-sparse vectors, the dimension d of

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those matrices is always linear in n. Specifically, one of the results of [6] guarantees existence of $n \times d$ integer sensing matrices A for ℓ-sparse vectors, $1 \leqslant \ell \leqslant n$, such that $|A| = k$ and $d = \Omega(\sqrt{k}n)$ (more precisely, $d/n = 1.2938$ when $k = 1$), and a result of [8] implies existence of such sensing matrices with

$$n < d \leqslant \max\left\{k+1, \frac{k^{\frac{m}{m-1}}}{2}\right\}.$$

In our construction, we pay the price of the sparsity level being lower for the reward of allowing larger d. We prove Theorem 1.1 in Section 3, where we discuss a deterministic construction of such matrices.

Our construction of sensing matrices is based on a particular geometric covering problem. The general problem of covering point-sets by hyperplanes in n-dimensional Euclidean spaces has been extensively studied by various authors. For instance, in [2] the authors present an overview of the previously known and the current state-of-the-art estimates on the number of linear and affine subspaces needed to cover lattice points in a given $\mathbf{0}$-symmetric convex body. On the other hand, [1] considers the problem of covering all but one of the vertices of an n-dimensional cube by the minimal possible number of hyperplanes, whereas the classical no-three-in-line problem asks for a maximal collection of points in a planar $T \times T$ integer grid so that no three of them lie on the same straight line. There is a number of other variations of such covering problems studied in discrete and convex geometry.

We discuss the problem of covering a set of points by parallel hyperplanes. Specifically, suppose $S \subset \mathbb{R}^n$ is a set of k points. It is not difficult to notice that S can always be covered by no more than $\max\{1, k-n+1\}$ parallel hyperplanes (Lemma 2.1). However, while sufficient, is this number necessary? In other words, does there exist a set of $k \geqslant n$ points in $\mathbb{R}^n$ that cannot be covered by fewer than $k-n+1$ parallel hyperplanes? This question arises naturally in connection with the famous Tarski plank problem, as we demonstrate in Section 4 below.

More specifically, one can ask for such a set on a lattice grid. Let $T \geqslant 1$ be an integer and let

$$C_n(T) := \{\boldsymbol{x} \in \mathbb{Z}^n : |\boldsymbol{x}| \leqslant T\}$$

be the integer cube of sidelength $2T$ centered at the origin in $\mathbb{R}^n$. Since every finite set of integer lattice points is contained in some such integer cube, we will consider specifically subsets of $C_n(T)$. First, notice that $2T+1$ parallel hyperplanes cover all of $C_n(T)$. We prove the following simple lemma in Section 2.

Lemma 1.2. *Let $S \subseteq C_n(T)$ be a set of points of cardinality k.*

1. *If no fewer than $k-n+1$ parallel hyperplanes can cover S, then $k \leqslant 2T+n$.*
2. *If $2T+1$ parallel hyperplanes are required to cover S, then $k \geqslant 2T+n$.*

These observations raise a natural question: does there exist a subset S of $C_n(T)$ of cardinality $2T+n$ that cannot be covered by fewer than $2T+1$ parallel hyperplanes? In Proposition 2.2 below we answer this questions in the affirmative, demonstrating constructions of such sets for the case $T = 1$; in other words, we show that for each $n \geqslant 1$ there exists a set $S_n \subset C_n(1)$ of cardinality $n+2$ which cannot be covered by fewer than 3 parallel hyperplanes.

We can now show how point-sets of cardinality k in $\mathbb{R}^n$ that cannot be covered by fewer than $k-n+1$ parallel hyperplanes can be used to construct sensing matrices for sparse signal recovery. For a set of k points $S = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_k\} \subset \mathbb{R}^n$ define a partition of S into two disjoint subsets

$$I_m = \{\boldsymbol{x}_{i_1}, \dots, \boldsymbol{x}_{i_m}\},\ J_l = \{\boldsymbol{x}_{j_1}, \dots, \boldsymbol{x}_{j_l}\} = S \backslash I_m, \tag{1}$$

so that $I_m \cap J_l = \varnothing$ and $S = I_m \cup J_l$, where $m, l \geqslant 1$ are such that $k = m + l$. For this partition, define the corresponding set of pairwise difference vectors

$$\mathcal{D}(I_m, J_l) = \{\boldsymbol{x}_i - \boldsymbol{x}_j : \boldsymbol{x}_i \in I_m, \boldsymbol{x}_j \in J_l\},$$

so $|\mathcal{D}(I_m, J_l)| \leqslant ml = m(k - m)$. For a subset $D \subseteq \mathcal{D}(I_m, J_l)$ define support of D to be the set of all distinct vectors $\boldsymbol{x}_i$ that appear in the differences in D. For instance, the support of the difference set

$$\{\boldsymbol{x}_1 - \boldsymbol{x}_2, \boldsymbol{x}_3 - \boldsymbol{x}_2, \boldsymbol{x}_1 - \boldsymbol{x}_4, \boldsymbol{x}_3 - \boldsymbol{x}_4\}$$

is $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4\}$. Let us write $A(D)$ for the matrix whose column vectors are elements of the set D. Define also a bipartite graph $\Gamma(D)$ with vertices corresponding to the support of D. Two vertices $\boldsymbol{x}_i, \boldsymbol{x}_j$ are then connected by an edge if and only if $\boldsymbol{x}_i - \boldsymbol{x}_j \in D$, in other words D is the set of edges of $\Gamma(D)$. We write $g(D)$ for the minimal length of a cycle in the graph $\Gamma(D)$ called the girth of this graph. We can now state the following result.

Theorem 1.3. *Let $S = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_k\} \subset \mathbb{R}^n$ be a collection of $k > n$ points, $m, l \geqslant 1$ integers such that $k = m + l$, $S = I_m \sqcup J_l$ partition of S, and $D \subseteq \mathcal{D}(I_m, J_l)$. Let $1 \leqslant \ell \leqslant n - 1$. The following two statements are true:*

1. *If S cannot be covered by fewer than $k - n + 1$ parallel hyperplanes and for every subset D' of ℓ vectors of D, $g(D') > \ell$, then $A(D)$ is a sensing matrix for ℓ-sparse vectors.*
2. *If for every $m + l = k$ and partition $S = I_m \sqcup J_l$, $A(D(I_m, J_l))$ is a sensing matrix for n-sparse vectors, then S cannot be covered by fewer than $k - n + 1$ parallel hyperplanes.*

Proof of Theorem 1.3 is presented in Section 3, where we also combine Proposition 2.2 and Theorem 1.3 to obtain a family of sensing matrices with good properties, hence proving Theorem 1.1. We are now ready to proceed.

2. Parallel hyperplane coverings

In this section we discuss the problem of covering point-sets with parallel hyperplanes. To start with, we can ask how many parallel hyperplanes are needed to cover a set of k points in $\mathbb{R}^n$?

Lemma 2.1. *If $S \subset \mathbb{R}^n$ is a set of cardinality k, then it can be covered by no more than* $\max\{1, k - n + 1\}$ *parallel hyperplanes.*

Proof. There is a unique hyperplane passing through every set of n points in general position in $\mathbb{R}^n$. If $k \leqslant n$ or S contains no more than n points in general position, then S is covered by one such hyperplane. If $k > n$ and $S = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_k\}$ contains some n points in general position, say $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$, then they determine a unique hyperplane H. Then there are at most $k - n$ remaining points $\boldsymbol{x}_{n+1}, \dots, \boldsymbol{x}_k$ in S which are not covered by H. Translating H along a line L orthogonal to H at most $k - n$ times we can cover these remaining points. Hence the total number of parallel hyperplanes required to cover S is $k - n + 1$. □

We also give a quick proof of Lemma 1.2.

Proof of Lemma 1.2. Taking $2T+1$ parallel translates of any coordinate hyperplane covers the entire integer cube $C_n(T)$. Therefore we automatically have

$$k - n + 1 \leqslant 2T + 1,$$

which means that $k \leqslant 2T + n$. This proves (1).

To prove (2), suppose $k < 2T+n$, and let S_1 be a subset of S containing $\min\{k, n\}$ points. Let H be a hyperplane through the points of S_1, then there are at most $2T - 1$ points of S not contained in H, and so at most $2T - 1$ parallel translates of H will cover these points. Hence a total of at most $2T$ hyperplanes is enough to cover S, which is a contradiction. Hence $k \geqslant 2T + n$, and every subset of $C_n(T)$ can be covered by $2T + 1$ parallel hyperplanes. □

We can now prove the existence of point-sets on $0, \pm 1$ grid in $\mathbb{R}^n$ that require a maximal covering, demonstrating an explicit construction of such sets for each n.

Proposition 2.2. *For each $n \geqslant 1$ there exists a set $S_n \subset C_n(1)$ of cardinality $n+2$ which cannot be covered by fewer than 3 parallel hyperplanes.*

Proof. Consider the set of $n + 2$ vectors

$$T_n = \{\boldsymbol{0}, \boldsymbol{e}_1, \dots, \boldsymbol{e}_n, \boldsymbol{z}_n\} \subset \mathbb{R}^n,$$

where $\boldsymbol{z}_n = -\sum_{i=1}^n \boldsymbol{e}_i$. Suppose that this set can be covered by just two parallel hyperplanes. This means that there exists a nonzero vector $\boldsymbol{v} \in \mathbb{R}^n$ so that the linear functional $\langle \boldsymbol{v}, \cdot \rangle$ attains only two distinct values on T_n. One of these values must be 0, since $\langle \boldsymbol{v}, \boldsymbol{0} \rangle = 0$. If $\boldsymbol{v}$ has all nonnegative coordinates, then $\langle \boldsymbol{v}, \boldsymbol{e}_i \rangle > 0$ while $\langle \boldsymbol{v}, \boldsymbol{z}_n \rangle < 0$; similarly, if $\boldsymbol{v}$ has all nonpositive coordinates, then $\langle \boldsymbol{v}, \boldsymbol{e}_i \rangle < 0$ while $\langle \boldsymbol{v}, \boldsymbol{z}_n \rangle > 0$. Then assume that $\boldsymbol{v}$ contains both, positive and negative coordinates, say $v_i > 0$ and $v_j < 0$ for some $1 \leqslant i \neq j \leqslant n$. In this case, $\langle \boldsymbol{v}, \boldsymbol{e}_i \rangle > 0$ while $\langle \boldsymbol{v}, \boldsymbol{e}_j \rangle < 0$. Hence T_n cannot be covered by just two parallel hyperplanes. □

Further, there are other possible sets satisfying the property of Proposition 2.2. Indeed, the simple construction we demonstrate in the proof above was suggested to us by an anonymous reviewer, while our original construction was different: for each $n \geqslant 1$, define $S_n = \{\boldsymbol{0}, \boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{n+1}\}$, where

$$\boldsymbol{x}_i = -\boldsymbol{e}_{n-i+1} + \sum_{j=n-i+2}^{n} \boldsymbol{e}_j \ \ \forall\ 1 \leqslant i \leqslant n,\ \ \boldsymbol{x}_{n+1} = (1, \dots, 1)^\top, \tag{2}$$

with $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n \in \mathbb{R}^n$ being the standard basis vectors. This set also has cardinality $n + 2$ and cannot be covered by any two parallel hyperplanes in $\mathbb{R}^n$, although our original proof was more complicated. In fact, these constructions can be generalized: one can take a set consisting of the origin together with the $n + 1$ vertices of any simplex containing the origin in its interior.

3. Sensing matrices

Here is our first observation on constructions of some sensing matrices.

Proposition 3.1. *Let $k > n$ and $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{k-1} \in \mathbb{R}^n$ be distinct nonzero vectors. Let*

$$S = \{\mathbf{0}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{k-1}\} \subset \mathbb{R}^n.$$

Let A be the $n \times (k-1)$ matrix, whose columns are these vectors, i.e.

$$A = \begin{pmatrix} \boldsymbol{x}_1 & \ldots & \boldsymbol{x}_{k-1} \end{pmatrix}.$$

If S cannot be covered by fewer than $k - n + 1$ parallel hyperplanes, then A is a sensing matrix for n-sparse signals.

Proof. Suppose that the number of distinct orthogonal projections of S onto every line is at least $k-n+1$. Arguing towards a contradiction, assume that some minor of A is zero. This means that the corresponding n vectors are linearly dependent, without loss of generality assume that these vectors are $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$. Hence they all lie in some subspace of dimension $m \leqslant n-1$, call this subspace V. Naturally, $\mathbf{0}$ also lies in V, since V is a subspace. If all of the points $\boldsymbol{x}_{n+1}, \ldots, \boldsymbol{x}_{k-1}$ also lie in some $(n-1)$-dimensional subspace V' containing V, then $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{k-1}$ all project to one point on the line orthogonal to V', which is a contradiction. Hence assume that

$$\operatorname{span}_{\mathbb{R}}\{V, \boldsymbol{x}_{n+1}, \ldots, \boldsymbol{x}_{k-1}\} = \mathbb{R}^n.$$

Then there exist some $(n-1) - m$ points among $\boldsymbol{x}_{n+1}, \ldots, \boldsymbol{x}_{k-1}$ which do not lie in V. Let V' be the $(n-1)$-dimensional subspace spanned by V and these points. This means that V' contains a total of

$$n + (n-1) - m + 1 \geqslant n + 1$$

points of the set S. Let L be the line through the origin orthogonal to V', then all of these points project to one point on L. Since the number of remaining points in our collection is $k - (n+1)$, the total number of distinct projections of points of S onto L is at most $k - n$, which is a contradiction. Thus all minors of A must be nonzero. □

Notice that a direct converse of Theorem 3.1 is not true. Consider, for example, the 2×4 matrix

$$A = \begin{pmatrix} 2 & 1 & 3 & 2 \\ 1 & 2 & 1 & 2 \end{pmatrix},$$

and let $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4$ be the column vectors of A. Let $S = \{\mathbf{0}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_4\}$, so $k = 5$. Then all minors of A are nonzero, hence A is a sensing matrix for 2-sparse signals, and $k - n + 1 = 4$. However, projections of these five points onto the line along the vector $(1,1)$ are the three points: $(0,0)$, $(3/\sqrt{2}, 3/\sqrt{2})$ and $(4/\sqrt{2}, 4/\sqrt{2})$. Hence these five points can be covered by three parallel lines orthogonal to this line.

In fact, if we are to use integer point sets like S in Theorem 3.1, then simultaneously achieving k much greater than n and $|A|$ small becomes difficult. This problem can be remedied by using difference sets at the expense of having ℓ, the sparsity level, smaller than n as in Theorem 1.3, which we now prove.

Proof of Theorem 1.3. First suppose that at least $k - n + 1$ parallel hyperplanes are required to cover S and $g(D) > \ell$. To prove that $A(D)$ is a sensing matrix for ℓ-sparse vectors, we simply need to establish that no ℓ vectors of D lie in the same $(\ell-1)$-dimensional subspace of $\mathbb{R}^n$. Suppose they do, say some ℓ vectors

$$\boldsymbol{y}_1 = \boldsymbol{x}_{i_1} - \boldsymbol{x}_{j_1}, \ldots, \boldsymbol{y}_\ell = \boldsymbol{x}_{i_\ell} - \boldsymbol{x}_{j_\ell} \tag{3}$$

are in the same $(\ell-1)$-dimensional subspace V, where $\boldsymbol{x}_{i_1},\dots,\boldsymbol{x}_{i_\ell} \in I_m$ and $\boldsymbol{x}_{j_1},\dots,\boldsymbol{x}_{j_\ell} \in J_l$. Assume that $s \geqslant 1$ out of the $\boldsymbol{x}_{i_u}$ vectors are distinct and $p \geqslant 1$ of the $\boldsymbol{x}_{j_u}$ vectors are distinct: let S_1 be the set of these $s+p$ distinct vectors. Without loss of generality assume that $s \leqslant p$. Let U be the $(n-\ell+1)$-dimensional subspace of $\mathbb{R}^n$ orthogonal to V, then each pair $\boldsymbol{x}_{i_r}, \boldsymbol{x}_{j_r}$ lies in the same parallel translate of V along U. So if, for instance, $\boldsymbol{x}_1-\boldsymbol{x}_2$, $\boldsymbol{x}_1-\boldsymbol{x}_3$ and $\boldsymbol{x}_4-\boldsymbol{x}_2$ are in V, then $\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4$ all must lie in the same parallel translate of V along U. Since the $s+p$ distinct vectors $\boldsymbol{x}_{i_r}, \boldsymbol{x}_{j_r}$ correspond to some vertices in the graph $\Gamma(D)$ and their ℓ difference vectors correspond to edges between these vertices, $s+p > \ell$: otherwise $\Gamma(D)$ would contain a cycle of length $\leqslant \ell$, contradicting the assumption that $g(D) > \ell$. The number of parallel translates of V along U needed to cover the set S_1 is at most

$$t := s - (\ell - p) \geqslant 1.$$

Let V_1 be the parallel translate of V along U containing the pair $\boldsymbol{x}_{i_1}, \boldsymbol{x}_{j_1}$. Since $k-n+1 \geqslant 2$, S cannot be covered completely by any single $(n-1)$-dimensional hyperplane containing V_1. Since dimension of V_1 is $\ell-1$, there must exist a set $Z \subset S \backslash V_1$ consisting of $n-\ell$ points in general position. Let H_1 be an $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$ through Z and V_1 and let $L \subset U$ be the line through the origin orthogonal to H_1. Let us write $Z = Z_1 \sqcup Z_2$, where $Z_1 = Z \cap S_1$: here it is possible for Z_1 or Z_2 to be empty. Then H_1 covers all the points of S_1 in V_1 plus at least $|Z_1|$ more, and so H_1 together with at most $t-|Z_1|-1$ additional parallel translates of H_1 along L cover S_1. Now at most $k-(s+p)-|Z_2|$ additional parallel translates of H_1 along L will cover the rest of S. Hence a total of at most

$$\begin{aligned} & (t-|Z_1|) + (k-(s+p)-|Z_2|) = t - |Z| + k - (s+p) \\ = \; & s - (\ell-p) - (n-\ell) + k - (s+p) = k-n < k-n+1 \end{aligned}$$

parallel hyperplanes covers S. This is a contradiction, and hence $A(D)$ is a sensing matrix for ℓ-sparse vectors.

In the opposite direction, suppose that every $A(I_m, J_l)$ is a sensing matrix for n-sparse vectors, so no n vectors in the set $D(I_m, J_l)$ are linearly dependent. Suppose S can be covered by some collection of $t \leqslant k-n$ parallel hyperplanes. Out of these hyperplanes, let $H_1,\dots,H_s$ be those that contain more than one point of S, then the remaining $t-s$ hyperplanes $H_{s+1},\dots,H_t$ (if any) contain just one point of S each, $1 \leqslant s \leqslant t$. Then

$$\left| S \cap \left(\bigcup_{i=1}^{s} H_i \right) \right| = k - (t-s) \geqslant k - (k-n-s) = n+s.$$

For each $1 \leqslant i \leqslant s$, let

$$S \cap H_i = \{\boldsymbol{x}_{i1},\dots,\boldsymbol{x}_{ij_i}\},$$

hence $\sum_{i=1}^{s} j_i \geqslant n+s$. Let I_t be the set consisting of all the vectors $\boldsymbol{x}_{i1}$ for $1 \leqslant i \leqslant s$, and all the vectors from $S \cap H_j$ for $s+1 \leqslant j \leqslant t$. Let $l = k-t$, and let $J_l = S \backslash I_t$. Consider the set of difference vectors

$$D' = \{\boldsymbol{x}_{i1}-\boldsymbol{x}_{i2},\dots,\boldsymbol{x}_{i1}-\boldsymbol{x}_{ij_i} : 1 \leqslant i \leqslant s\} \subseteq D(I_t, J_l).$$

Since all of the vectors $\boldsymbol{x}_{i1},\dots,\boldsymbol{x}_{ij_i}$, $1 \leqslant i \leqslant s$ lie in parallel hyperplanes, all the vectors of D' lie in the same $(n-1)$-dimensional subspace of $\mathbb{R}^n$. The total number of these vectors is

$$|D'| = \sum_{i=1}^{s}(j_i - 1) \geqslant n + s - s = n,$$

hence they are linearly dependent. This is a contradiction, hence S cannot be covered by any collection of fewer than $k - n + 1$ parallel hyperplanes. □

Finally, we turn to Theorem 1.1. Viewing our setup in terms of the bipartite graph $\Gamma(D)$, we are interested in having the girth $g(D) \geqslant \ell + 1$ with D, the set of edges of $\Gamma(D)$ as large as possible. The problem of constructing such graphs has been extensively studied by various authors, see [14] for a survey of known results in this direction. In particular, Theorem 3 of [14] guarantees that for large enough k, there exist such graphs with k vertices and

$$\geqslant \left(\frac{k}{2}\right)^{1+\frac{2}{3\ell-2}} \tag{4}$$

edges. An explicit deterministic construction of such bipartite graphs was carried out in [10] and [9] (also see [11]). We can now use this result to prove our theorem.

Proof of Theorem 1.1. For sufficiently large n, let S_n be the set of $n + 2$ vectors with $\{0, \pm 1\}$ coordinates obtained in Proposition 2.2, hence S_n cannot be covered by $(n+2)-n+1 = 3$ parallel hyperplanes. Let Γ be a bipartite graph on the $n+2$ vertices corresponding to the vectors of S_n with the number of edges satisfying (4). Let D be the set of difference vectors corresponding to the edges of Γ, then $g(D) > \ell$. Therefore by Theorem 1.3, $A(D)$ is a sensing matrix for ℓ-sparse vectors, and we have $|A(D)| = 2$. Furthermore, $A(D)$ is an $n \times d$ integer matrix where

$$d \geqslant \left(\frac{n+2}{2}\right)^{1+\frac{2}{3\ell-2}},$$

by (4). Notice that if $\ell = o(\log n)$, then $d/n \to \infty$ as $n \to \infty$, meaning that d is bigger than linear in n. □

Example 1. *Consider the set S_3 as given in (2). Partitioning it into the first three vectors and the remaining two, compute the difference set D corresponding to the complete $(3,2)$-bipartite graph Γ. Then*

$$A(D) = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 & -2 & -2 \\ -1 & -1 & -2 & -2 & 0 & 0 \end{pmatrix}$$

is a 3×6 sensing matrix for 3-sparse vectors, since Γ does not have any 3-cycles.

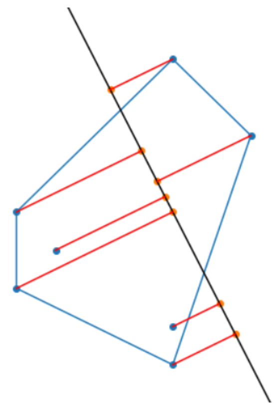

FIGURE 1. Projecting a point-set onto a line

4. Covering convex sets by parallel planks

In this section, we discuss a connection of our hyperplane covering questions with the classical Tarski plank problem. Let M be a nonempty convex compact set in $\mathbb{R}^n$. Its width $w(M)$ is defined to be the smallest distance between two parallel supporting hyperplanes to M. A plank P in $\mathbb{R}^n$ is a strip of space between two parallel hyperplanes and its width $h(P)$ is the distance between them. The classical conjecture of Tarski [13], proved by Bang [3], [4] asserts that if a finite collection of planks $P_1, \dots, P_k$ covers M then

$$\sum_{i=1}^{k} h(P_i) \geqslant w(M).$$

The simplest such covering is by parallel planks. But what if we want to cover M by a collection of parallel planks which misses a prescribed collection of points inside of M: how wide can such planks be? We prove the following observation.

Proposition 4.1. *Let M be a compact convex set of width w in $\mathbb{R}^n$ and let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k$ be points in M. Let S be a collection of all planks P in $\mathbb{R}^n$ with both bounding hyperplanes intersecting M so that the interior of P does not contain any point $\boldsymbol{x}_i$. For each such plank write $h(P)$ for its width, and define*

$$H := \sup_{P \in S} h(P).$$

Then

$$H \geqslant \begin{cases} \frac{w}{k-n+2} & \text{if } k \geqslant n \\ \frac{w}{2} & \text{if } k < n. \end{cases} \tag{5}$$

To prove this proposition, we need a lemma, which is interesting in its own right.

Lemma 4.2. *Let $k, n \geqslant 1$ be integers and $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k \in \mathbb{R}^n$ be distinct points, and let C be their convex hull. Let w be the width of C, and let L be a line in $\mathbb{R}^n$. Let $\boldsymbol{y}_1, \dots, \boldsymbol{y}_m$ be distinct projections of $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k$ onto L, $2 \leqslant m \leqslant k$ (see Figure 1). Then*

$$\max_{1 \leqslant i \leqslant m} \min_{1 \leqslant j \leqslant m} \left\{ \|\boldsymbol{y}_i - \boldsymbol{y}_j\| : j \neq i \right\} \geqslant \frac{w}{m-1}. \tag{6}$$

In other words, the maximal gap between consecutive projection points $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$ *along* L *is at least* $w/(m-1)$. *Furthermore, there exists a line* L *with*

$$m \leqslant \max\{1, k-n+1\}. \tag{7}$$

Proof. Let V be a hyperplane orthogonal to L positioned so that it does not intersect C. Start moving V towards C by translating along the line L until it meets the first point $\boldsymbol{x}_i$: call this translated hyperplane V_1. Continue translating V further along L until it meets the next point $\boldsymbol{x}_j$ not contained in V_1: call this translated hyperplane V_2. Continue translating in this manner until all of the points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ are covered by the union of these hyperplanes. Notice that each of these hyperplanes are orthogonal to L, and hence project to one of the points $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$ on L: this means that there are precisely m such hyperplanes, $V_1, \ldots, V_m$ (without loss of generality, let us reindex the points $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$ so that V_i projects to $\boldsymbol{y}_i$). For each $1 \leqslant i \leqslant m-1$, define P_i to be the plank (that is, a strip of space between two parallel hyperplanes) bounded by V_i and V_{i+1}, then the width h_i of P_i is precisely $\|\boldsymbol{y}_{i+1} - \boldsymbol{y}_i\|$, the gap between consecutive points on L, i.e.

$$h_i = \min_{1 \leqslant j \leqslant m} \left\{ \|\boldsymbol{y}_i - \boldsymbol{y}_j\| : j \neq i \right\}. \tag{8}$$

Notice that $P_1, \ldots, P_{m-1}$ are parallel planks intersecting only in the boundary, the union of which covers C. Then Bang's solution to the Tarski Plank Problem [3], [4] implies that

$$\sum_{i=1}^{m-1} h_i \geqslant w. \tag{9}$$

Now (6) follows from (9) combined with (8).

To establish (7), notice that we can always pick a hyperplane containing at least n of the points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$: every collection of n points in $\mathbb{R}^n$ lies in a hyperplane, and this hyperplane is determined uniquely if the points in question are in general position. Let L be the line orthogonal to this hyperplane. If $k \leqslant n$, then all the points are in this hyperplane and hence project to one point on L. Then assume that $k > n$. Following the procedure described above with respect to this choice of the line L, we see that at least one of the hyperplanes V_i will contain at least n points out of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$. Assuming that each next one contains only one of the remaining points, the total resulting number of hyperplanes will be $\leqslant k-n+1$: this is precisely the number of projection points $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$ onto L. This gives (7). □

Remark 4.1. Notice that in Lemma 4.2 we assume that the number of projections $m \geqslant 2$. Indeed, if $m = 1$ then there are no pairs of points, and thus no gaps between them.

Proof of Proposition 4.1. Let C be the convex hull of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$. Arguing as in the proof of Lemma 4.2 above, let L be a line with the number m of projection points $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$ on it minimized. Let U_1, U_2 be parallel hyperplanes, orthogonal to L and tangent to M so that M is contained between them, then distance between them is $\geqslant w$. Now let us start building planks, as before. Let V_1 be a translate of U_1 along L in the direction of M which contains the closest to U_1 point $\boldsymbol{x}_i$, and continue these translations the same way as above until we reach U_2. The total number of hyperplanes we construct this way will be at most $m + 2$, and hence they define at most $m+1$ parallel planks that cover M and do not contain any of the points $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k$ in their interiors. The maximal width of such a plank is $\geqslant w/(m+1)$, which is $\geqslant w/(k-n+2)$ by (7), unless $n < k$: in this last case, only two planks are needed, since all the points will be contained in one hyperplane. □

The bound of Proposition 4.1 is optimal. Take, for instance, $n = 2$ and let M be an equilateral triangle with height equal to 2, then $w(M) = 2$. Let $k = 3$ and $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in M$ be vertices of a scaled-down equilateral triangle $M' = \frac{1}{3}M$ with height equal to $2/3$ centered at the center of M. The set of planks S as in the statement of Proposition 4.1 contains a plank P whose one boundary line passes through vertices $\boldsymbol{x}_1, \boldsymbol{x}_2$ and the other through $\boldsymbol{x}_3$. Then

$$h(P) = w(M') = \frac{2}{3} = \frac{w(M)}{k-n+2},$$

which is precisely the lower bound of the proposition. Notice that no plank in S can have width greater than $h(P)$, and thus the lower bound of (5) when $k \geqslant n$ is achieved. In the situation $k < n$, we can take, for example, $n = 2$ again, M a unit disk, $k = 1$ and $\boldsymbol{x}_1 =$ center of M. Then a plank P bounded by a line through $\boldsymbol{x}_1$ and a parallel line tangent to the boundary circle of M has maximal possible width of all planks in S, and

$$h(P) = 1 = \frac{w(M)}{2},$$

again achieving the lower bound of (5) in this case.

More generally, one can ask if there exists a points set $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k \in \mathbb{R}^n$ so that the number of their distinct projections onto every line L in $\mathbb{R}^n$ is at least $k-n+1$? Explicit examples of such sets for $k = n+2$ were demonstrated in Section 2 above.

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