reasonable one. Here we show by a quite different argument that if a rational form f is isotropic then there is always a solution of $f(\mathbf{a}) = 0$ which is not too big.

LEMMA 8.1. *Let*

$$f(\mathbf{x}) = \Sigma f_{ij} x_i x_j \in \mathbf{Z}[x_1, \ldots, x_n] \tag{8.1}$$

be an isotropic form in n variables. Then there is an

$$\mathbf{a} \in \mathbf{Z}^n, \mathbf{a} \neq \mathbf{0} \tag{8.2}$$

with

$$f(\mathbf{a}) = 0 \tag{8.3}$$

such that

$$\max_{1 \leqslant j \leqslant n} |a_j| \leqslant (3F)^{(n-1)/2}, \tag{8.4}$$

where

$$F = \sum_{i,j} |f_{ij}|. \tag{8.5}$$

Here $|\ |$ denotes the absolute value.

Note. Although the constant 3 in (8.4) could doubtless be improved, the exponent $(n-1)/2$ cannot. For let b be a large positive integer and consider

$$f(\mathbf{x}) = x_1^2 - \sum_{j=2}^{n} (x_j - bx_{j-1})^2. \tag{8.6}$$

Here

$$F = n + 2(n-1)\,b + (n-1)\,b^2. \tag{8.7}$$

Suppose that $f(\mathbf{a}) = 0$. Then clearly $a_1 \neq 0$ and

$$a_n = \lambda_n + \lambda_{n-1} b + \ldots + \lambda_2 b^{n-2} + a_1 b^{n-1}, \tag{8.8}$$

where

$$\lambda_n = a_n - ba_{n-1},$$

so

$$\lambda_n^2 + \lambda_{n-1}^2 + \ldots + \lambda_2^2 = a_1^2. \tag{8.9}$$

It is easy to check that these equations imply that

$$\begin{aligned} |a_n| &\geqslant -|a_1| b^{n-2} + |a_1| b^{n-1} \\ &\geqslant b^{n-1} - b^{n-2}. \end{aligned} \tag{8.10}$$

Proof of Lemma. We take for $\mathbf{a} \in \mathbf{Z}^n, \mathbf{a} \neq \mathbf{0}$ a solution of $f(\mathbf{a}) = 0$ for which

$$\|\mathbf{a}\| \text{ (say)} = \max_{1 \leqslant j \leqslant n} |a_j| \tag{8.11}$$

is minimal. If (8.4) is false we shall find an $\mathbf{a}^* \in \mathbf{Z}^n, \neq \mathbf{0}$ for which $f(\mathbf{a}^*) = 0$, $\|\mathbf{a}^*\| < \|\mathbf{a}\|$: which would contradict the minimality of $\|\mathbf{a}\|$.

We may suppose by permuting the indices and taking $-\mathbf{a}$ for $\mathbf{a}$ if need be that

$$a_1 = \max_j |a_j|. \tag{8.12}$$

If $a_1 = 1$ there is nothing to prove, so we suppose that

$$a_1 \geqslant 2. \tag{8.13}$$

Let $\theta_2, \ldots, \theta_n$ be any real numbers. By a theorem on diophantine approximation (Theorem 2.4, Corollary, of Chapter 5) there are integers $b_1, \ldots, b_n$ such that

$$0 < b_1 < a_1 \tag{8.14}$$

and

$$|b_1\theta_j - b_j| \leqslant a_1^{-1/(n-1)} \qquad (2 \leqslant j \leqslant n). \tag{8.15}$$

We apply this with

$$\theta_j = a_j/a_1. \tag{8.16}$$

Then

$$\begin{aligned} |b_j| &\leqslant |b_1\theta_j| + a_1^{-1/(n-1)} \\ &\leqslant b_1 + a_1^{-1/(n-1)} \\ &< b_1 + 1; \end{aligned}$$

and so

$$\|\mathbf{b}\| = \max_j |b_j| = b_1 < \|\mathbf{a}\|. \tag{8.17}$$

The minimality of $\|\mathbf{a}\|$ now implies that

$$f(\mathbf{b}) \neq 0. \tag{8.18}$$

We now choose $\lambda, \mu \in \mathbf{Z}$ so that

$$\mathbf{a}^* = \lambda\mathbf{a} + \mu\mathbf{b} \tag{8.19}$$

satisfies

$$f(\mathbf{a}^*) = 0. \tag{8.20}$$

We have

$$\begin{aligned} f(\mathbf{a}^*) &= \lambda^2 f(\mathbf{a}) + 2\lambda\mu f(\mathbf{a}, \mathbf{b}) + \mu^2 f(\mathbf{b}) \\ &= 2\lambda\mu f(\mathbf{a}, \mathbf{b}) + \mu^2 f(\mathbf{b}); \end{aligned}$$

so it is enough to choose

$$\lambda = f(\mathbf{b}) \in \mathbf{Z}\,(\neq 0) \tag{8.21}$$

and

$$\mu = -2f(\mathbf{a}, \mathbf{b}) \in \mathbf{Z}. \tag{8.22}$$

We note that

$$\mathbf{a}^* \neq \mathbf{0} \tag{8.23}$$

since otherwise (8.19) would imply $f(\mathbf{b}) = f(\mathbf{a}) = 0$ contrary to (8.18).

By (8.16) we can write (8.15) in the shape

$$b_j = \phi a_j + \delta_j$$

where

$$\phi = b_1/a_1$$

and

$$\delta_1 = 0, \qquad |\delta_j| \leqslant a_1^{-1/(n-1)} \qquad (2 \leqslant j \leqslant n). \tag{8.24}$$

We shall express $\mathbf{a}^*$ in terms of a and $\boldsymbol{\delta}$ by eliminating $\mathbf{b}$.

We have

$$f(\mathbf{a}, \mathbf{b}) = f(\mathbf{a}, \phi\mathbf{a} + \boldsymbol{\delta}) = \phi f(\mathbf{a}) + f(\mathbf{a}, \boldsymbol{\delta}) = f(\mathbf{a}, \boldsymbol{\delta})$$

and

$$f(\mathbf{b}) = f(\phi\mathbf{a} + \boldsymbol{\delta}) = 2\phi f(\mathbf{a}, \boldsymbol{\delta}) + f(\boldsymbol{\delta}).$$

Hence by (8.19), (8.21), (8.22) we have

$$\begin{aligned} \mathbf{a}^* &= f(\mathbf{b})\mathbf{a} - 2f(\mathbf{a}, \mathbf{b})\mathbf{b} \\ &= \{2\phi f(\mathbf{a}, \boldsymbol{\delta}) + f(\boldsymbol{\delta})\}\,\mathbf{a} - 2f(\mathbf{a}, \boldsymbol{\delta})\{\phi\mathbf{a} + \boldsymbol{\delta}\} \\ &= f(\boldsymbol{\delta})\,\mathbf{a} - 2f(\mathbf{a}, \boldsymbol{\delta})\,\boldsymbol{\delta}. \end{aligned}$$

Estimating crudely and recalling the definition (8.5) of F it follows that

$$\|\mathbf{a}^*\| \leqslant 3F\|\mathbf{a}\| \; \|\delta\|^2.$$

By the minimality of $\|\mathbf{a}\|$ we have, however,

$$\|\mathbf{a}^*\| \geqslant \|\mathbf{a}\|$$

and so

$$3F\|\delta\|^2 \geqslant 1. \tag{8.25}$$

But by (8.24) we have

$$\|\delta\| \leqslant a_1^{-1/(n-1)} = \|\mathbf{a}\|^{-1/(n-1)}.$$

Hence (8.25) implies

$$\|\mathbf{a}\| \leqslant (3F)^{(n-1)/2},$$

as asserted.

9. AN APPROXIMATION THEOREM

The object of this section is to prove

LEMMA 9.1. *Let $f(\mathbf{x})$ be an isotropic form over $\mathbf{Q}$ in $n \geqslant 3$ variables. Let $\varepsilon > 0$ be arbitrarily small and P be a finite set of primes p (possibly $\infty \in P$) and for $p \in P$ let $\mathbf{b}_p \in \mathbf{Q}_p^n$ be given with $f(\mathbf{b}_p) = 0$. Then there is a $\mathbf{b} \in \mathbf{Q}^n$*