The Urysohn universal metric space and hyperconvexity

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Abstract. In this paper we prove that Urysohn univeral space is hyperconvex. We also examine the Gromov hyperbolicity and hyperconvexity of metric spaces. Using four-point property, we give a proof of the fact that hyperconvex hull of a δ-Gromov hyperbolic space is also δ-Gromov hyperbolic.

1 Introduction

In a paper published posthumously[25], Pavel Samuilovich Urysohn constructed a complete, separable metric space $\mathbb{U}$ that contains an isometric copy of every complete separable metric space, for an obvious reason $\mathbb{U}$ is called *universal*. It is well known that every separable metric space X isometrically embeds in a Banach space l^∞ (Fréchet embedding) and a theorem of Banach [4] provides a concrete separable target for all separable spaces, namely every separable metric space embeds isometrically in $\mathcal{C}[0,1]$, here $\mathcal{C}[0,1]$ is the separable Banach space of continuous real-valued functions on the closed unit interval equipped with the sup norm. However, the interest of the Urysohn space $\mathbb{U}$ does not lie in its universality alone, it has the following property:

Proposition 1.1. *Let $\mathbb{U}$ be a separable and complete metric space that contains an isometric image of every separable metric space. Then $\mathbb{U}$ is Urysohn universal if and only if $\mathbb{U}$ has the following finite transitivity property: every isometry between finite subsets of $\mathbb{U}$ extends to an isometry of $\mathbb{U}$ onto itself.*

Urysohn in [25] also proved that $\mathbb{U}$ exists and moreover, up to isometry there is only one such space. It is worth remarking that the Banach space $\mathcal{C}[0,1]$ cannot be Urysohn universal since every isometric bijection between Banach spaces is an affine map (see [5], page 341), thus Proposition 1.1 is not satisfied. Construction of Urysohn's universal space is published in full details in [25], and for the sake of completeness and accessibility we shall briefly go through the construction.

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Urysohn's construction

Urysohn first constructs a countable metric space $\mathbb{U}_0$ containing the image of every countable metric space for which the distance between any two points is rational. The completion $\mathbb{U}$ of $\mathbb{U}_0$, (i.e., the unique complete metric space that contain $\mathbb{U}_0$ as a dense subset) is the Urysohn universal space. We now proceed to the construction of the space $\mathbb{U}_0$. To define a convenient metric on $\mathbb{U}_0$, Urysohn first considers all systems consisting of a finite number of positive rationals and denotes the collection of all these systems by $\mathbb{Q}$ and labels each individual system as $\mathbb{Q}_i$, $i \in \mathbb{N}$. First, consider all systems of $\mathbb{Q}$ that are composed of only one rational, order them arbitrarly, and assign to them in order natural indices that are not divisible by 4. Now consider all systems of $\mathbb{Q}$ composed of $p > 1$ rationals, and number them using all natural numbers divisible by 2^p, but not divisible by 2^{p+1}. Every system $\mathbb{Q}$ receives, in this way, a unique number, and generate the ordering

$$\mathbb{Q}_1, \mathbb{Q}_2, \cdots, \mathbb{Q}_n, \cdots,$$

each $\mathbb{Q}_n$ being a system, is composed of some finite number p_n of positive rationals, and can be written in the form

$$\mathbb{Q}_n = [r_1^{(n)}, r_2^{(n)}, \cdots, r_{p_n}^{(n)}]$$

where $r_1^{(n)}, r_2^{(n)}, \cdots r_{p_n}^{(n)}$ are the rational elements of $\mathbb{Q}_n$. Note that $p_1 = 1$ and p_n is the cardinality of $\mathbb{Q}_n$ and we have for all $n > 1$, $n > p_n$. Let $\mathbb{U}_0$=$\{a_1, a_2, \cdots, a_n, \cdots\}$ be the metric space with its metric defined in the following way. We begin by setting $\rho(a_1, a_1) = 0$. Suppose for all $i, k \leq n+1$, the nonnegative value $\rho(a_i, a_k)$ is given. For the sake of simplicity assume that $p_n = q$ and $r_i^{(n)} = \rho_i$ for $i \leq q$ and consider the following two cases:

a) At least one of the inequalities

$$|\rho_i - \rho_k| \leq \rho(a_i, a_k) \leq \rho_i + \rho_k \text{ where } i, k \leq q \tag{1.1}$$

is not satisfied. Then, we say that the system $\mathbb{Q}_n$ is *incorrectly defined* and set

$$\rho(a_{n+1}, a_j) = \max_{i,k \leq q} \rho(a_i, a_k),$$

for all $j \leq n$.

b) In the case of all of inequalities in 1.1 satisfied, we say $\mathbb{Q}_n$ is *correctly defined* and set

$$\rho(a_{n+1}, a_j) = \min_{\lambda \leq q} \{\rho(a_j, a_\lambda) + \rho_\lambda\}, \text{ for all } j \leq n.$$

The triangle inequality for ρ is proved by induction according its definition. Note that the value of metric in the second case is defined so that the triangle inequality trivially holds, and the first case is crucial for the whole construction. It follows that $\mathbb{U}_0$ is a universal space for all countable metric spaces having rationals for values of their metrics. Next, Urysohn extends the above procedure to the completion $\mathbb{U}$ of $\mathbb{U}_0$, which implies as a consequence that $\mathbb{U}$ is universal for all separable metric spaces. The following theorem is due to Uryshon [25] which will be used throughout this paper.

Theorem 1.2. *For any finite subset $x_1, x_2, \cdots, x_n$ of $\mathbb{U}$ and positive real numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that $|\alpha_i - \alpha_j| \leq \rho(x_i, x_j) \leq \alpha_i + \alpha_j$ one can find $y \in \mathbb{U}$ such that $\rho(y, x_i) = \alpha_i$ for every $i = 1, 2, \cdots, n$.*

We now give the definition of hyperconvexity.

Definition 1.1. A metric space (X, d) is said to be *hyperconvex* if $\bigcap_{i \epsilon I} B(x_i; r_i) \neq \phi$ for every collection $B(x_i; r_i)$ of closed balls in X for which $d(x_i, x_j) \leq r_i + r_j$, where by $B(x; r)$ we mean the *closed* ball centered at x with radius $r \geq 0$.

This notion was first introduced by Aronszajn and Panitchpakdi in [3], where it was shown that a metric space is hyperconvex if and only if it is injective with respect to nonexpansive (1-Lipschitz) mappings. Later Isbell [20] showed that every metric space has an injective hull, therefore every metric space is isometric to a subspace of a minimal hyperconvex space. Hyperconvex metric spaces are complete and connected [1]. The simplest examples of hyperconvex spaces are the set of real numbers $\mathbb{R}$, or a finite-dimensional real Banach space endowed with the maximum norm. While the Hilbert space l_2 fails to be hyperconvex, the spaces L^∞ and l_∞ are hyperconvex. In [1] it is also shown that there is a general "linking construction" yielding hyperconvex spaces. Moreover, in these spaces paths between points are restricted; they must pass through certain "common" points. On the other hand, the concept of a metric tree in graph theory also has a built-in restriction. A complete metric space X is a *metric tree* provided that for any two points x and y in X there is a unique arc joining x and y, and this arc is a geodesic arc. The study of metric trees, also known as T-theory or R-trees began with J. Tits [24] in 1977 and since then, applications have been found for metric trees within many fields of mathematics.(see [2] For an overview of geometry, topology, and group theory applications, consult Bestvina [6]). A complete discussion of these spaces and their relation to $CAT(0)$ spaces we refer the reader to well known monograph of Bridson and Haefliger [9].

Definition 1.2. A metric space (X, d) is called $\delta - $*hyperbolic* for $\delta \geq 0$ if for each $x, y, z, p \in X,\ d(x, y) + d(z, p) \leq \max\{d(x, z) + d(y, p), d(x, p) + d(y, z)\} + 2\delta.$

Definition 1.2 is a generalization of famous four-point property for which $\delta = 0$. Four-point property plays an important role in metric trees, for example, in [1], it is shown that a metric space is a metric tree if and only if it is complete, connected and satisfies the four-point property. However, it is also well known that a complete geodesic metric space X is a CAT(0) if and only if it satisfies four-point condition (see [9]). Furthermore, recall that we call X *hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$. Sometimes δ is referred as a *hyperbolicity constant* for X. Besides any tree being 0-hyperbolic, any space of finite diameter, δ, is δ-hyperbolic and the hyperbolic plane $\mathbb{H}^2$ is $(\frac{1}{2}\log 3)$-hyperbolic. In fact any simply connected Riemanian manifold with curvature bounded above by some negative constant $-\kappa^2 < 0$ is $(\frac{1}{2\kappa}\log 3)$-hyperbolic (see[9]). For more on hyperbolic spaces we refer the reader to [15],[26],[8],[18] and [19].

Isbell's Hyperconvex Hull

Isbell in [20] introduced injective envelope of a metric space. In the following we give some of Isbell's ideas. Let M be a metric space and for any $x \in M$, define $f_x : M \to [0, \infty)$ by $f_x(y) = d(x, y)$, using triangle inequality one can easily show (see [14]) that for any $x, y, a \in M$ we have

$$d(x, y) \leq f_a(x) + f_a(y) \quad \text{and} \quad f_a(x) \leq d(x, y) + f_a(y).$$

Furthermore, if we let $f : M \to [0, \infty)$ be such that $d(x, y) \leq f(x) + f(y)$ for any $x, y \in M$, and if $f(x) \leq f_a(x)$ for all $x \in M$ and some $a \in M$ then $f_a = f$.

Definition 1.3. Let A be any subset of M, we say $f : A \to [0, \infty)$ is *extremal* if $d(x, y) \leq f(x) + f(y)$ for all $x, y \in A$ and is pointwise minimal, i.e., if $g : A \to [0, \infty)$ is another function such that $d(x, y) \leq g(x) + g(y)$ for all $x, y \in A$, and if $g(x) \leq f(x)$ for all $x \in A$, then we must have $f = g$.

It is not difficult to see that every extremal function is non-negative and 1-Lipschitz.

Definition 1.4. Let $A \subset (M, d)$, *the injective envelope* of A, denoted by $h(A)$ is the set of all extremal functions defined on A. In other words,

$$h(A) = \{f : A \to [0, \infty) : \ d(x, y) \leq f(x) + f(y) \ \text{ and } f \text{ is minimal}\}$$

$\rho(f, g) = \sup_{x \in A} d(f(x), g(x))$ defines a metric on $h(A)$.

The map $e : A \to h(A)$ defined as $e(a) = f_a$ is an isometry because

$$\rho(e(a), e(b)) = \sup_{x \in A} |f_a(x) - f_b(x)| = \sup_{x \in A} |d(a, x) - d(b, x)| = d(a, b).$$

Thus, we can identify A with the subspace $e(A)$ of $h(A)$. Furthermore, we have the following extension property [20]:
Let A be a nonempty subset of M and $r : A \to [0,\infty)$ be such that $d(x,y) \leq r(x) + r(y)$ for all $x, y \in A$. Then there exist $R : M \to [0,\infty)$ which extends r and $d(x,y) \leq R(x) + R(y)$ for all $x, y \in M$. Furthermore, there exist an extremal function f defined on M such that $f(x) \leq R(x)$ for all $x \in M$. It is also worth noting that one can easily show

$$|f(x) - f(y)| \leq d(x,y)$$

for any $x, y \in A$. This implies that $h(A) \subset Lip_1(A)$.
Aronszajn and Panitpacti in [3] showed that:

a) A metric space M is injective if and only if it is an absolute 1-Lipschitz retract.

b) A metric space M is injective if and only if it is hyperconvex.

In the following we list some properties of $h(A)$, which we will use in the proof of Theorem 2.4. For the proof of the following properties we refer the reader to [14].

Proposition 1.3. *a) If $f \in h(A)$, then it satisfies $f(x) \leq d(x,y) + f(y)$ for all $x, y \in A$. Moreover*

$$f(x) = \sup_y |f(y) - f_x(y)| = \rho(f, e(x)).$$

b) If $f \in h(A)$, then it satisfies $f(x) = \sup_{z \in A} \{d(x,z) - f(z)\}$.

c) if A is compact, then $h(A)$ is compact.

d) $h(A)$ is hyperconvex.

e) If $A \subset B \subset h(A)$, then $h(B)$ is isometric to $h(A)$.

Note that every subset of a hyperconvex space has a hyperconvex hull and all hyperconvex hulls are related with isometries. Following is a well known example which shows that hyperconvex hull of a set does not have to be unique.

Example 1.4. *Consider the hyperconvex space $\mathbb{R}^2$ with the maximum norm and take $A = \{(0,0),(0,1)\}$. Then the sets*

$$h_1(A) = \{(x,y) \in \mathbb{R}^2 : \ x = y,\ 0 \leq x \leq 1\}$$

and

$$h_2(A) = \{(x,y) \in \mathbb{R}^2 : \ x = y,\ 0 \leq x \leq 1/2\} \cup \{(x,y) \in \mathbb{R}^2 : \ x = 1 - y,\ 1/2 \leq x \leq 1\}$$

are both hyperconvex hulls of A.

It is worth mentioning that H. Herrlich in [17] proved that the hyperconvex hull of $\mathbb{R}^n$ with the sup metric d_1 is $\mathbb{R}^{2n-1}$ supplied with the maximum metric d_∞ and ℓ_∞ the space of all realvalued bounded sequences with the sup metric d_∞, is the hyperconvex hull of its subspace c_0, consisting of all sequences which converge to zero. Furthermore, H.B. Cohen in [11] proved the existence and uniqueness of an injective envelope for any Banach (normed) space over $\mathbb{R}$ and showed that an injective Banach space is linearly isometric to a function space $C(M)$, where M is compact Hausdorff and extremally disconnected. For more on hyperconvex hulls see [12] and [13]. In particular, for more on hyperconvex hulls of a normed space see [23].

Remark 1.5. There is lots of similarities in the steps of construction which yields to Urysohn universal space or to hyperconvex hull. If we consider **finite dimensional** subsets of a metric space and follow the construction of Isbell's hyperconvex hull one reaches spaces $E(X)$ (see page 20 of [16]), where $E(X)$ is the collection of functions $f : X \to \mathbb{R}$ that satisfy both

$$|f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y)$$

whenever $x, y \in X$ and

$$f(x) = \inf\{d(x, y) + f(y) : \ y \in Y\}$$

for all $x \in X$, where $Y \subset X$ is a finite set, called a support of f. If we consider by $E_n(X)$ the subspace of $E(X)$ consisting of those functions in $E(X)$ that have a support of cardinality at most n, then we have

$$X \subset E_1(X) \subset E_2(X) \cdots ,$$

and

$$E(X) = \bigcup_{n=1}^{\infty} E_n(X).$$

Thus, if we start with an arbitrary **separable** metric space X and define by induction

$$X_0 := X, \ \ X_{n+1} := E(X_n), \ \ n = 0, 1, 2, \ldots$$

then it can be shown that the space

$$X_\infty := \bigcup_{n=0}^{\infty} X_n$$

is Urysohn universal. (see [16]).

2 Main Results

Definition 2.1. A metric space (X, d) is said to be *metrically convex* if for any $x, y \in X$ and for any $r, s > 0$

$$\overline{B}(x, s) \cap \overline{B}(y, t) \neq \emptyset \quad \text{if and only if} \quad d(x, y) \leq s + t.$$

Lemma 2.1. *The Urysohn universal space* $(\mathbb{U}, \rho)$ *is metrically convex.*

Proof. Let $x_1, x_2 \in \mathbb{U}$ and $r_1, r_2 > 0$ be given. Clearly, $\overline{B}(x_1, r_1) \cap \overline{B}(x_2, r_2) \neq \emptyset$ implies $\rho(x_1, x_2) \leq r_1 + r_2$ by the triangle inequality. Thus we only need to show the reverse implication. Assume that $\rho(x_1, x_2) \leq r_1 + r_2$. We consider the following cases:

a) Suppose $r_1 \leq \rho(x_1, x_2) + r_2$ and $r_2 \leq \rho(x_1, x_2) + r_1$ holds true. Then by Theorem 1.2 there exist $y \in \mathbb{U}$ such that $\rho(y, x_i) = r_i$ for $i = 1, 2$. thus $y \in B(x_1, r_1) \cap B(x_2, r_2)$, so $\bigcap_{i=1}^{2} B(x_i, r_i) \neq \emptyset$.

b) Either $r_1 > \rho(x_1, x_2) + r_2$ or $r_2 > \rho(x_1, x_2) + r_1$ holds true. Without loss of generality assume $r_1 > \rho(x_1, x_2) + r_2$. Then, for any $y \in B(x_2, r_2)$ we have

$$\rho(y, x_1) \leq \rho(y, x_2) + \rho(x_2, x_1) \leq r_2 + r_1 - r_2 = r_1.$$

Hence $y \in B(x_1, r_1)$, implying $B(x_2, r_2) \subset B(x_1, r_1)$ and $\bigcap_{i=1}^{2} B(x_i, r_i) \neq \emptyset$.

□

Definition 2.2. We say a metric space (X, d) satisfies *the finite ball intersection property* (FBIP) if for any $x_1, x_2, \cdots, x_n \in X$ and $r_1, r_2, \cdots r_n > 0$ with $d(x_i, x_j) \leq r_i + r_j$ implies $\bigcap_{i=1}^{n} \overline{B}(x_i, r_i) \neq \emptyset$

Lemma 2.2. *The Urysohn universal space* $(\mathbb{U}, \rho)$ *satisfies FBIP.*

Proof. Suppose $x_1, x_2, \cdots, x_n \in X$ and $r_1, r_2, \cdots r_n > 0$ are given so that $d(x_i, x_j) \leq r_i + r_j$

a) Suppose $|r_i - r_j| \leq \rho(x_i, x_j)$ is true. Then by the Theorem 1.2 of Urysohn, there exists $y \in \mathbb{U}$ such that $\rho(y, x_i) = r_i$ hence $y \in \bigcap_{i=1}^{n} \overline{B}(x_i, r_i) \neq \emptyset$.

b) Assume that there exist i_0, j_0 such that $|r_{i_0} - r_{j_0}| > \rho(x_{i_0}, x_{j_0})$, then $r_{i_0} > \rho(x_{i_0}, x_{j_0}) + r_{j_0}$ or $r_{j_0} > \rho(x_{i_0}, x_{j_0}) + r_{i_0}$, without loss of generality assume $r_{i_0} > \rho(x_{i_0}, x_{j_0}) + r_{j_0}$, thus as in case b) of proof of Lemma 2.1, we obtain $B(x_{j_0}, r_{j_0}) \subset B(x_{i_0}, r_{i_0})$. Therefore, it is enough to show that

$$\bigcap_{\{1,2,\cdots,n\}\setminus\{i_0\}} B(x_i, r_i) \neq \emptyset$$

given that $\rho(x_i, x_j) \leq r_i + r_j$ for all $i, j \in \{1, 2, \cdots, n\} \setminus \{i_0\}$.

□

Theorem 2.3. *The Urysohn universal space* $(\mathbb{U}, \rho)$ *is hyperconvex.*

Proof. We need to show that for any family $\{x_\alpha\}$ of points in $\mathbb{U}$ and any family of positive numbers $\{r_\alpha\}$ with $\rho(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ the intersection $\bigcap_\alpha \overline{B}(x_\alpha, r_\beta)$ is nonempty. If the collection $\{x_\alpha\}$ is finite the claim follows from Lemma 2.2. Suppose $\{x_\alpha\}$ is countable. By re-labeling we set

$$\{x_1, x_2, \cdots, x_n, \cdots\} \text{ and } \{r_1, r_2, \cdots, r_n, \cdots\}.$$

Fix $n \geq 1$ and consider the finite subcollection $\{x_1, x_2, \cdots, x_n\}$. By Lemma 2.2 $\exists y_n \in \mathbb{U}$ such that $\rho(y_n, x_k) = r_k$ for $\forall k = 1, 2, \cdots, n$. In particular $y_n \in \bigcap_{k=1}^{n} \overline{B}(x_k, r_k)$. Now consider the sequence $\{y_n\}$ in $\mathbb{U}$. By definition of y_n, $y_n \in \overline{B}(x_1, r_1)$. Since $\mathbb{U}$ is complete and $\overline{B}(x_1, r_1)$ is closed $\overline{B}(x_1, r_1)$ is compact. Therefore, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ that converges to a point $y \in \overline{B}(x_1, r_1)$, we claim that $y \in \bigcup_{n=1}^{\infty} \overline{B}(x_n, r_n)$. Suppose $y \notin \bigcup_{n=1}^{\infty} \overline{B}(x_n, r_n)$. Then $\exists n_0$ such that $y \notin \overline{B}(x_{n_0}, r_{n_0})$ since $\overline{B}(x_{n_0}, r_{n_0})$ is closed, $\exists \epsilon > 0$ such that $B(y, \epsilon) \cap \overline{B}(x_{n_0}, r_{n_0}) = \emptyset$. Since $\lim_{k\to\infty} y_{n_k} = y$, $\exists n_k \geq n_0$ such that $y_{n_k} \in B(y, \epsilon)$. Hence $y_{n_k} \notin \overline{B}(x_{n_0}, t_{n_0})$. But by definition $y_{n_k} \in \bigcap_{m=1}^{n_k} \overline{B}(x_m, r_m)$ and in particular $y_{n_k} \in \overline{B}(x_{n_0}, r_{n_0})$ since $n_0 \leq n_k$, a contradiction. □

The injective hull of hyperbolic groups are studied in [21] and [22]. The following theorem shows that if X is a δ-hyperbolic space then $h(X)$ is also an δ-hyperbolic space. Although such a result is appeared in [21], our proof has a different approach and perhaps sheds more light onto the four point condition and its generalization to define δ-hyperbolic spaces.

Theorem 2.4. *If (X,d) is δ-hyperbolic, then $h(X)$ is also δ-hyperbolic.*

Proof. We first show that if a metric space (X,d) is δ-hyperbolic and $f \in h(X)$, then $X \cup \{f\}$ is δ-hyperbolic too. Suppose $f, x, y, z \in X \cup \{f\}$, then

$$d(x,y) + \rho(h_z, f) = d(x,y) + f(z) = \sup_{p \in X}\{d(x,y) + d(z,p) - f(p)\}$$

which implies

$$d(x,y)+\rho(h_z,f) \leq \max\{\sup_{p\in X}\{d(x,z)+d(y,p)-f(p)\}, \sup_{p\in X}\{d(x,p)+d(y,z)-f(p)\}\}+2\delta.$$

Hence

$$d(x,y) + \rho(h_z, f) \leq \max\{d(x,z) + f(y), d(y,z) + f(x)\} + 2\delta$$

and

$$d(x,y) + \rho(h_z, f) \leq \max d(x,z) + \rho(h_y, f), d(y,z) + \rho(h_x, f)\} + 2\delta.$$

This proves that $X \cup \{f\}$ is δ-hyperbolic. Since

$$X \subset X \cup \{f\} \subset h(X)$$

using the above property e) of the Proposition 1.3, we deduce that $h(X \cup \{f\}) = h(X)$. Setting $f = f_1$ and using the argument above, by taking $f_2 \in h(X \cup \{f_1\})$, we can show that $X \cup \{f_1, f_2\}$ is δ-hyperbolic. Continuing in this manner and adding one point at a time concludes the proof. □

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