MATH 195, SPRING 2019, TAKE-HOME MIDTERM

Please print your name clearly!

Name:____

This test is due on Thursday, 3/28/19, in class. While completing it, feel free to use your lecture notes from class, as well as those posted on the class webpage. You are however not allowed to consult with anyone: it is understood that solutions to this midterm represent solely your own work with no outside assistance. Good luck!

Problem 1. (10 points) Let Λ be a lattice of rank r in $\mathbb{R}^n$, $1 \leq r < n$, and let $V = \operatorname{span}_{\mathbb{R}} \Lambda$. Prove that Λ is a discrete co-compact subgroup of V.

Problem 2. (15 points) Let Λ be a lattice of full rank in $\mathbb{R}^n$, $n \geq 1$. Let M be a compact convex $\mathbf{0}$-symmetric set in $\mathbb{R}^n$ with $\mathrm{Vol}(M) > 0$. Define a counting function

$$f(t) = |tM \cap \Lambda|,$$

i.e. $f(t)$ gives the number of points of Λ in the homogeneous expansion of M by t. Prove that

$$\lim_{t\to\infty} \frac{f(t)}{t^n} = \frac{\mathrm{Vol}(M)}{\det(\Lambda)}.$$

Problem 3. (15 points) Let a, b be positive real numbers, and suppose that

$$\Lambda = \begin{pmatrix} a & b & 0 & 0 \\ 0 & 1 & a & b \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \mathbb{Z}^4$$

is a full-rank sublattice of $\mathbb{Z}^4$.

a) - (5 points) What are all the possible values of a and b? Prove your answer.

b) - (5 points) Suppose that $a = b$ and Ω is a full-rank sublattice of Λ, such that the volume of a fundamental domain of Ω in $\mathbb{R}^4$ is equal to 20. What are all the possible values of a? Prove your answer.

c) - (5 points) Assuming part b, can Λ be a sublattice of any of the following two lattices:

$$L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{pmatrix} \mathbb{Z}^4, \ L_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 3 \end{pmatrix} \mathbb{Z}^4,$$

and if so, which one(s)? What are all the possible values of a in each case? Prove your answers.

Problem 4. (10 points) Let $n \geq 2$ be even, and define quadratic forms

$$Q_1(X_1, \dots, X_n) = \sum_{i=1}^{n} X_i^2 - X_1X_2 - X_3X_4 - \cdots - X_{n-1}X_n,$$

and

$$Q_2(X_1, \dots, X_n) = \boldsymbol{X}^t B \boldsymbol{X}$$

for some real symmetric matrix B. Suppose that Q_1 and Q_2 are isometric.

a) - (5 points) Find eigenvalues of B. Prove your answer.

b) - (5 points) Let

$$Q_1'(X_1, \dots, X_n) = \frac{1}{2}\sum_{i=1}^{n} X_i^2 - X_1X_2 - X_3X_4 - \cdots - X_{n-1}X_n.$$

Are Q_1' and Q_2 isometric? Prove your answer.

Problem 5. (10 points) Let $n \geq 2$, and let $F : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that

(1) $F(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$,
(2) $F(a\boldsymbol{x}) = |a|F(\boldsymbol{x})$ for all $a \in \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^n$.

Let

$$X = \{\boldsymbol{x} \in \mathbb{R}^n : F(\boldsymbol{x}) \leq 1\}.$$

Assume additionally that F satisfies the triangle inequality:

$$F(\boldsymbol{x} + \boldsymbol{y}) \leq F(\boldsymbol{x}) + F(\boldsymbol{y}), \tag{1}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Let Λ be a lattice of full rank in $\mathbb{R}^n$. Prove that for every real number

$$\mu \geq 2\left(\frac{\det(\Lambda)}{\mathrm{Vol}(X)}\right)^{1/n}$$

the intersection $\mu X \cap \Lambda$ contains a non-zero vector. Is this statement still true if F does not satisfy the triangle inequality (1)? Either prove your answer or give a counter-example.

Problem 6. (10 points) Prove that the hexagonal lattice is perfect and eutactic. Further, prove that a lattice L in $\mathbb{R}^2$ is perfect if and only if it is similar to the hexagonal lattice.

Problem 7. (10 points) Let Λ be a lattice of full rank in $\mathbb{R}^n$, and let $M \subset \mathbb{R}^n$ be a compact convex set such that $\mathrm{Vol}(M) < \det(\Lambda)$. Prove that there exists a point $\boldsymbol{x} \in \mathbb{R}^n$ such that the intersection $M \cap (\Lambda + \boldsymbol{x})$ is empty.

Problem 8. (10 points) Let $\{n_j\}_{j=1}^{\infty}$ be a sequence of natural numbers such that

$$\lim_{j\to\infty} \frac{n_{j+1}}{n_j} = \infty.$$

Define

$$f(x) = \sum_{j=1}^{\infty} x^{n_j},$$

and let α be a real algebraic number such that $0 < \alpha < 1$. Prove that $f(\alpha)$ is transcendental.