MATH 195, SPRING 2019, TAKE-HOME FINAL

Please print your name clearly!

Name:____

This test is due on Tuesday, 5/7/19, in class. While completing it, feel free to use your lecture notes from class, as well as those posted on the class webpage. You are however not allowed to consult with anyone: it is understood that solutions to this midterm represent solely your own work with no outside assistance. Additionally, if you use a homework exercise, please include its solution. Good luck!

Problem 1. (40 points) Consider the number field $K = \mathbb{Q}(\sqrt{3}, \sqrt{-3})$.

Part a (10 points) Determine the degree of K over $\mathbb{Q}$. Prove your answer.

Part b (10 points) Find a primitive element for K over $\mathbb{Q}$. Prove your answer.

Part c (10 points) Find the minimal polynomial of the primitive element you found in part b. Prove your answer.

Part d (10 points) Is 2 a prime in K? Prove your answer.

Problem 2. (10 points) Let $K = \mathbb{Q}(\sqrt{-13})$. Is $\mathcal{O}_K$ a PID (principal ideal domain)? Prove your answer.

Problem 3. (30 points) Let $\alpha, \beta \in \mathbb{C}$, both nonzero, let $\gamma = \alpha/\beta$, and suppose $\mathbb{Q}(\alpha) \cong \mathbb{Q}(\beta)$.

Part a (10 points) Give necessary and sufficient conditions on α and β so that $[\mathbb{Q}(\gamma) : \mathbb{Q}] < \infty$. Prove your answer.

Part b (10 points) Suppose α and β are algebraic and $\gamma \in \mathbb{Q}$. Let

$$f(x) = \sum_{k=0} a_k x^k$$

be the minimal polynomial of α. Determine the minimal polynomial $g(x)$ of β.

Part c (10 points) Suppose α and β are algebraic and $f(x) = g(x)$, i.e. they have the same minimal polynomial. Is it true that $\gamma \in \mathbb{Q}$? Prove or give a counterexample.

Problem 4. (20 points) Is the sequence $\{\cos n\}_{n=0}^{\infty}$ dense in the interval $[-1, 1]$? Prove your answer.