MATH 30 - 5, FALL 2021, PRACTICE TEST II

Disclaimer: **This practice test DOES NOT serve as an indication of the contents of the actual test. It only suggests a possible format.**

Please print your name clearly!

Name: **SOLUTIONS**

Please show all your work, that is explain every step of your solution - it is your work, not the answer, that is being evaluated. When asked to prove a statement, make sure to provide reasoning behind each claim you are making in the process of the proof. The use of calculators or any other electronic devices is prohibited during the test. You are also not allowed to use any study materials except for those provided to you during the test. Cheating is strictly prohibited, and will be prosecuted. Good luck!

Problem 1 (40 points). Compute the following limits or prove they do not exist. Show all your work.

a) - (10 points) $\lim_{x\to-\infty}\left(x+\sqrt{x^2+2x}\right)$

b) - (10 points) $\lim_{x\to 0^+}\tan^{-1}(\ln x)$

c) - (10 points) $\lim_{x\to\infty}(e^{-x}+\sin x)$

d) - (10 points) $\lim_{x\to\infty}\frac{x^6+1}{1+x^2-x^6}$

Solution. a) Notice that

$$\begin{aligned}\lim_{x\to-\infty}\left(x+\sqrt{x^2+2x}\right) &= \lim_{x\to-\infty}\frac{\left(x+\sqrt{x^2+2x}\right)\left(x-\sqrt{x^2+2x}\right)}{x-\sqrt{x^2+2x}}\\ &= \lim_{x\to-\infty}\frac{x^2-(x^2+2x)}{x-\sqrt{x^2+2x}}\\ &= -2\lim_{x\to-\infty}\frac{x}{x-\sqrt{x^2+2x}} = -2\lim_{x\to-\infty}\frac{1}{1-\sqrt{1+\frac{2}{x}}}.\end{aligned}$$

Now observe that as $x \to -\infty$, $2/x$ is negative, but tends to 0, i.e. it tends to 0 from below. Hence $\sqrt{1+2/x}$ tends to 1 from below, and so the denominator $1 - \sqrt{1+\frac{2}{x}}$ tends to 0 from above. Therefore

$$\lim_{x\to-\infty} \frac{1}{1-\sqrt{1+\frac{2}{x}}} = \infty,$$

hence

$$\lim_{x\to-\infty} \left(x + \sqrt{x^2+2x}\right) = -2 \lim_{x\to-\infty} \frac{1}{1-\sqrt{1+\frac{2}{x}}} = -\infty.$$

b) As $x \to 0^+$, $\ln x \to -\infty$, and so $\tan^{-1}(\ln x) \to -\pi/2$, since as the argument tends to $-\pi/2$ its tangent tends to $-\infty$. Therefore:

$$\lim_{x\to 0^+} \tan^{-1}(\ln x) = -\frac{\pi}{2}.$$

c) As $x \to \infty$, $e^{-x} \to 0$, but $\sin x$ keeps fluctuating in the interval $[-1, 1]$. Therefore this limit does not exist.

d) Notice that with the standard trick of dividing the numerator and the denominator by the highest power of x present, we obtain:

$$\lim_{x\to\infty} \frac{x^6+1}{1+x^2-x^6} = \lim_{x\to\infty} \frac{1+\frac{1}{x^6}}{\frac{1}{x^6}+\frac{1}{x^4}-1} = -1,$$

since $1/x^4$ and $1/x^6$ tend to 0 as $x \to \infty$.

Problem 2 (30 points). Determine where are the following functions continuous.

a) - (10 points) $f(x) = \frac{x-2}{x^2+2}$.

b) - (10 points) $g(x) = \frac{\sqrt{x}}{\sin(x)}$.

c) - (10 points) $h(x) = \begin{cases} 3x+2 & \text{if } x < 2 \\ x^3 & \text{if } 2 \leq x \leq 3 \\ 3x^2 + x - 1 & \text{if } x > 3 \end{cases}$

Solution. a) This function is continuous everywhere, since numerator and denominator are polynomials (hence continuous everywhere), and the denominator is never equal to 0:

$$x^2 + 2 \geq 2 > 0$$

for all real values of x.

b) Notice that $\sqrt{x}$ is defined only when $x \geq 0$, and then it is continuous; $\sin x$ is continuous everywhere, but $\sin x =$ when $x = n\pi$ for any integer n. Therefore $g(x)$ is continuous when

$$x > 0,\ x \neq n\pi \text{ for any integer } n.$$

c) The function $h(x)$ is given by polynomial expressions, which are always continuous, hence the only points where $h(x)$ could be discontinuous are the joints of the piecewise definition $x = 2$ and $x = 3$. On the other hand,

$$\begin{aligned} \lim_{x\to 2^-} h(x) &= \lim_{x\to 2^-} (3x+2) = 3 \cdot 2 + 2 = 8 \\ &= 2^3 = \lim_{x\to 2^+} x^3 = \lim_{x\to 2^+} h(x) \\ &= h(2) = 2^3, \end{aligned}$$

so $h(x)$ is continuous at $x = 2$. However, $h(3) = 3^3 = 27 = \lim_{x\to 3^-} h(x)$, while

$$\lim_{x\to 3^+} h(x) = \lim_{x\to 3^+} (3x^2 + x - 1) = 3 \cdot 3^2 + 3 - 1 = 29,$$

and so $h(x)$ is not continuous at $x = 3$. Therefore $h(x)$ is continuous at all $x \neq 3$.

Problem 3 (20 points). Let

$$f(x) = \sqrt{x}$$

throughout this problem.

a) - (10 points) Use the limit definition of the derivative to find the function $f'(x)$ and specify its domain.

b) - (10 points) Find an equation for the tangent line to the graph of $y = f(x)$ at $x = 4$.

Solution. a) Observe that

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

Domain of $f'(x)$ is then given by all $x > 0$.

b) Equation of the tangent line to the graph of $f(x)$ at $x = a$ is given by

$$y = f'(a)x + (f(a) - f'(a)a).$$

We have $a = 4$, hence

$$f'(a) = \frac{1}{2\sqrt{4}} = \frac{1}{4}, \ f(a) = \sqrt{4} = 2.$$

Hence the tangent line is given by the equation

$$y = \frac{1}{4}x + \left(2 - \frac{1}{4} \times 2\right) = \frac{x}{4} + \frac{3}{2}.$$

Problem 4 (20 points). Find derivatives of the following functions.

a) - (10 points) $f(x) = e^x(x^3 - x^2 + x)$
b) - (10 points) $g(x) = \frac{e^x}{x^3 - x^2 + x}$

Solution. a) We use product rule to obtain:

$$\begin{aligned} f'(x) &= (e^x)'(x^3 - x^2 + x) + e^x(x^3 - x^2 + x)' \\ &= e^x(x^3 - x^2 + x) + e^x(3x^2 - 2x + 1) \\ &= e^x(x^3 + 2x^2 - x + 1). \end{aligned}$$

b) We use quotient rule to obtain:

$$\begin{aligned} g'(x) &= \frac{(e^x)'(x^3 - x^2 + x) - e^x(x^3 - x^2 + x)'}{(x^3 - x^2 + x)^2} \\ &= \frac{e^x(x^3 - x^2 + x) - e^x(3x^2 - 2x + 1)}{(x^3 - x^2 + x)^2} \\ &= \frac{e^x(x^3 - 4x^2 + 3x - 1)}{(x^3 - x^2 + x)^2}. \end{aligned}$$

Problem 5 (10 points). Let

$$f(x) = \frac{\sin x}{x}.$$

This function is continuous everywhere except at $x = 0$. Use the fact that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

to define a function $g(x)$ that is continuous everywhere and is equal to $f(x)$ at all $x \neq 0$. Is this function differentiable at $x = 0$? If so, use the limit definition of the derivative to compute $g'(0)$.

Hint: $\lim_{x \to 0} \frac{\sin x - x}{x^2} = 0$.

Solution. Define

$$g(x) = \begin{cases} f(x) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0, \end{cases}$$

then we have

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1 = g(0),$$

hence $g(x)$ is continuous at $x = 0$. When $x \neq 0$, $g(x) = f(x)$, which is continuous, hence $g(x)$ is continuous everywhere.

Now, the derivative of $g(x)$ at $x = 0$ is defined as

$$g'(0) = \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{\frac{\sin(0+h)}{0+h} - 1}{h} = \lim_{h \to 0} \frac{\sin h - h}{h^2} = 0.$$

Hence $g(x)$ is differentiable at $x = 0$ and $g'(0) = 0$.