MATH 172, FALL 2017, TAKE-HOME MIDTERM

Please print your name clearly!

Name:____

This test is due on Wednesday, 10/25/17, in class. While completing it, feel free to use your lecture notes from class, as well as your textbook. You are however not allowed to consult with anyone: it is understood that solutions to this midterm represent solely your own work with no outside assistance. Good luck!

Problem 1. (30 points) Let R be an integral domain and F its fraction field. R is called a *valuation domain* if for every nonzero element $\alpha \in F$, either $\alpha \in R$ or $\alpha^{-1} \in R$. Throughout this problem, assume that R is a valuation domain.

Part a (10 points). Let I and J be ideals in R. Prove that either $I \subseteq J$ or $J \subseteq I$. Conclude that R has a unique maximal ideal (recall that rings with this property are called local rings).

Part b (10 points). Suppose an ideal I in R is finitely generated. Prove that I must be principal.

Part c (10 points). Let $0 \neq \alpha \in F$ be such that

$$\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0 \in R$$

for some $a_{n-1}, \ldots, a_1, a_0 \in R$, $n \geq 1$. Prove that $\alpha \in R$.

Problem 2. (40 points) Let $R = \mathbb{Z}[x]$, the ring of polynomials in one variable x with integer coefficients.

Part a (20 points). Prove that maximal ideals in R are of the form $M = \langle p, f(x) \rangle$, where $p \in \mathbb{Z}$ is a prime number and $f(x) \in R$ is a polynomial, which is irreducible in $(\mathbb{Z}/p\mathbb{Z})[x]$.

Part b (10 points). Prove that the ideal $I = \langle 3, x^2 + x + 1 \rangle$ in R is not prime. Since maximal ideals in R are prime, I must not satisfy the description in part a. Explain how.

Part c (10 points). Is every nonzero prime ideal in R maximal? If yes, prove your answer; if no, give a counterexample.

Problem 3. **(30 points)** Let R be an integral domain and let $I_1, I_2, \ldots$ be a sequence of ideals in R.

Part a (10 points). Suppose that for some N,

$$I_1 \cap I_2 \cap \cdots \cap I_N = \{0\}.$$

Prove that at least one of I_n for $1 \leq n \leq N$ must be $\{0\}$.

Part b (10 points). Give an example of R and a sequence of ideals such that

$$\bigcap_{n=1}^{\infty} I_n = \{0\}$$

while none of the ideals I_n is equal to $\{0\}$.

Part c (10 points). Assume that

$$\bigcap_{n=1}^{\infty} I_n = \{0\}$$

in R and $I_n \neq \{0\}$ for any $n \geq 1$. Then prove that for every $m \geq 1$,

$$\bigcap_{n=m}^{\infty} I_n = \{0\}.$$

Problem 4. (40 points) Let R be a commutative ring with identity. R is called *Noetherian* if every ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$$

stabilizes, i.e. there exists some $n \geq 1$ such that $I_n = I_{n+1} = I_{n+2} = \dots$ R is called *Artinian* if every descending chain of ideals

$$\dots \subseteq I_3 \subseteq I_2 \subseteq I_1$$

stabilizes.

Part a (10 points). Is $\mathbb{Z}$ a Noetherian ring? Is it Artinian? Prove your answers without relying on the later parts of this problem.

Part b (10 points). Prove that an Artinian integral domain must be a field.

Part c (10 points). Define the *length* of a chain of $n+1$ prime ideals

$$P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \dots \subsetneq P_n$$

in a commutative ring R to be n (i.e., length is the number of inclusions instead of the number of ideals). *Krull dimension* of R, denoted $\dim(R)$, is the supremum of lengths of all chains of prime ideals in R. Prove that a commutative Artinian ring R must have $\dim(R) = 0$.

Part d (10 points). Suppose R is a PID, but not a field. Prove that R is Noetherian and $\dim(R) = 1$. Conclude that it cannot be Artinian.

Problem 5 (15 points). Let $D \in \mathbb{Z}$ be squarefree, and recall the definition of the corresponding quadratic integer ring:

$$\mathbb{Z}[\sqrt{D}] = \left\{ a + b\sqrt{D} : a, b \in \mathbb{Z} \right\},$$

and let the norm on it be given by the field norm on $\mathbb{Q}(\sqrt{D})$, as usual:

$$N(a + b\sqrt{D}) = \left| (a + b\sqrt{D})(a - b\sqrt{D}) \right| = \left| a^2 - Db^2 \right|,$$

for each $a+b\sqrt{D} \in \mathbb{Z}[\sqrt{D}]$. Recall also that this norm is multiplicative:

$$N(xy) = N(x)N(y),\ \forall\ x, y \in \mathbb{Z}[\sqrt{D}].$$

Part a (10 points). Prove that an element $x \in \mathbb{Z}[\sqrt{D}]$ is a unit if and only if $N(x) = 1$. Use this fact to prove that the only units in $\mathbb{Z}[i]$ are $\pm 1, \pm i$.

Part b (5 points). Suppose $n > 1$ is an integer such that $n = a^2 + b^2$ for some $a, b \in \mathbb{Z}$. Prove that n is not prime in $\mathbb{Z}[i]$.

Remark: Notice that there can be a prime $p \in \mathbb{Z}$ such that $p = a^2 + b^2$ for some $a, b \in \mathbb{Z}$ (for instance, $5 = 2^2 + 1^2$), but according to Part b it is no longer prime in $\mathbb{Z}[i]$. This effect is called *splitting* of a prime in an extension ring.

Problem 6 (10 points).

Part a (5 points). Let F be a field, and let $p(x) \in F[x]$ be an irreducible polynomial of degree 2. We know that the extension field K of F defined by $K = F[x]/\langle p(x) \rangle$ contains at least one root of $p(x)$. Prove that in fact it contains all roots of $p(x)$.

Part b (5 points). Prove that the polynomial

$$p(x) = 12x^8 + 15x^7 + 21x^5 + 6x^4 - 18x^3 + 2x^2 + 4x + 1$$

is irreducible in $\mathbb{Q}[x]$.

Problem 7 (10 points). Let R be an integral domain, and define $R[x, y]$ be the ring of polynomials in two variables x and y with coefficients in R. For which choices of R is $R[x, y]$ a Euclidean domain? Prove your answer.