MATH 172, FALL 2017, FINAL EXAM - TAKE HOME

Please print your name clearly!

Name:______________________________________

This test is due on Wednesday, 12/6/17, in class. While completing it, feel free to use your lecture notes from class, as well as your textbook. You are however not allowed to consult with anyone: it is understood that solutions to this exam represent solely your own work with no outside assistance. Good luck!

Problem 1 (15 points). Let F be a perfect field, and let $p(x) \in F[x]$ be an irreducible polynomial of degree 37. Suppose that $K := F[x]/\langle p(x)\rangle$ is the splitting field of $p(x)$. Prove that for every subfield B of K which contains F, the extension B/F is Galois.

Problem 2 (25 points). Let $f(x) = x^3 - 7$.

Part a - (15 points). Let K be the splitting field of $f(x)$ over $\mathbb{Q}$. Describe the Galois group of $K/\mathbb{Q}$, and describe all the intermediate fields E, i.e. subfields of K containing $\mathbb{Q}$. Which of them are Galois and which are not?

Part b - (10 points). Let L be the splitting field of $f(x)$ over $\mathbb{R}$. Describe the Galois group of $L/\mathbb{R}$.

Problem 3 (15 points).

Part a - (5 points). Describe all maximal ideals in $\mathbb{C}[x]$.

Part b - (5 points). Describe all maximal ideals in $\mathbb{R}[x]$.

Part c - (5 points). Describe (up to isomorphism) the fields that may be obtained as quotients $\mathbb{R}[x]/M$, where M is a maximal ideal in $\mathbb{R}[x]$.

Problem 4 (5 points). Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree 3, and G its Galois group. Prove that if G has order 3, then $f(x)$ splits completely over $\mathbb{R}$.

Problem 5 (20 points). Let R be a PID and F its field of fractions. Suppose S is a ring such that $R \subset S \subset F$.

Part a - (10 points). Prove that all elements of S can be written as a/b with $a, b \in R$ and $1/b \in S$.

Part b - (10 points). Prove that S is a PID.

Problem 6 (35 points). Let K be the splitting field of $x^4 - x^2 - 1$ over $\mathbb{Q}$.

Part a - (15 points). Find the Galois group of K/F.

Part a - (20 points). Give example of two subfields A and B of K so that $A/\mathbb{Q}$ is Galois and $B/\mathbb{Q}$ is not. Compute the Galois group of $A/\mathbb{Q}$.

Problem 7 (40 points). *Remark: This problem is closely connected to a problem we have seen on the midterm, however goes in the reverse direction.*

A PID R is called a *discrete valuation ring* (DVR) if it has precisely one non-zero prime ideal. Throughout this problem, let R be a DVR, P its unique prime ideal, and $K = \text{frac}(R)$ its field of fractions.

Part a - (10 points). Show that for any non-zero element $x \in K$ either $x \in R$ or $x^{-1} \in R$.

Part b - (10 points). Show that $\bigcap_{n\geq 1} P^n = 0$, where

$$P^n = \{a_1 a_2 \dots a_n : a_i \in P \; \forall \; 1 \leq i \leq n\}.$$

Part c - (10 points). Show that if $f(x) \in R[x]$ is a monic polynomial which has a root $a \in K$, then $a \in R$. Rings having this property are called *integrally closed.*

Part d - (10 points). Prove that K cannot be algebraically closed.

Problem 8 (15 points). Let R be a UFD.

Part a - (5 points). Prove that $R[x]$ is a UFD (you can use Gauss' Lemma and the fact that $F[x]$ is a PID when F is a field without proving them).

Part b - (10 points). Suppose that for any $a, b \in R$ the ideal $\langle a, b\rangle$ is principal. Prove that R is a PID.