PRACTICE MIDTERM SOLUTIONS (MATH 115A)

Problem 1. Label the following statements as true or false. You will receive 4 points for each correct answer, -4 for each incorrect one, and 0 if you give no answer.

(1) The union of two subspaces of a vector space is a subspace.
(2) A subset of a linearly independent set is linearly independent.
(3) For any $x_1, x_2 \in V$ and $y_1, y_2 \in W$ there is a linear transformation $T : V \to W$ such that $T(x_1) = y_1$, $T(x_2) = y_2$.
(4) For a matrix A the condition $A^3 = O$ implies that $A = O$. (Here O is the zero matrix).
(5) Two vector spaces of different dimensions can not be isomorphic.

Solution. (1) is false. For example, consider the x and y axes in $\mathbb{R}^2$. Each is a subspace, but their union is not (it is not closed under addition: $(1,0) + (0,1) = (1,1)$)

(2) is true. If there is a non-trivial linear combination equal to zero and involving the elements of a subset, then it is of course also a non-trivial linear combination equal to zero and involving elements of the set itself.

(3) is false. For example, if $y_1 \neq 0$ and $x_1 = 0$, there can be no linear T with $T(x_1) = y_1$, since $T(0) = 0$ by linearity.

(4) is false. For example, let A be the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $A \neq O$, but $A^2 = O$ and so is $A^3 = O$.

(5) is true. Any isomorphism takes a basis to a basis and hence preserves dimension.

Problem 2. Let $V = \mathbb{R}^n$, and let $T : V \to V$ be a map. Consider the subset of $\mathbb{R}^{2n}$ defined by

$$G = \{(v, w) : v, w \in \mathbb{R}^n, v = T(w)\}.$$

(This subset is called the *graph* of T). Show that T is linear if and only if G is a subspace of $\mathbb{R}^{2n}$.

Solution. Assume first that T is linear. We'll show that G is a subspace of $\mathbb{R}^{2n}$. To do this, we must prove that G is closed under addition and scalar multiplication. Let $x \in G$, $x' \in G$, $\alpha \in \mathbb{R}$. Thus x is a pair (v, w), so that $v, w \in \mathbb{R}^n$ and $v = T(w)$, and $y = (v', w')$ with $v' = Tw'$. Then

$$x + x' = (v + v', w + w').$$

Since $v + v' = T(w) + T(w') = T(w + w')$ by the linearity of T, it follows that $x + x' \in G$. Similarly,

$$\alpha x = (\alpha v, \alpha w).$$

Since $\alpha v = \alpha T(w) = T(\alpha w)$ by the linearity of T, we conclude that $\alpha x \in G$. Thus G is a vector subspace.

Assume now that G is a subspace. We must prove that T is linear. Let $w, w' \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. We must prove that $T(w + w') = T(w) + T(w')$ and that $T(\alpha w) = \alpha T(w)$. To prove this, note that by the definition of G, $(T(w), w) \in G$ and $(T(w'), w') \in G$. Since G is a vector subspace, also $(T(w), w) + (T(w'), w') = (T(w) + T(w'), w + w') \in G$ and $\alpha(T(w), w) = (\alpha T(w), \alpha w) \in G$. Using the definition of G, we learn that $T(w) + T(w') = T(w + w')$ and $\alpha T(w) = T(\alpha w)$.

Problem 3. Let u, v, w be three distinct vectors in a vector space V. Show that if $\{u, v, w\}$ is a basis, then so is the set $\{u+v+w, v+w, w\}$.

Solution. It is sufficient to prove that $\operatorname{Span}(u + v + w, v + w, w)$ is all of V. Indeed, since a basis of V has 3 elements, any set of 3 elements of V which spans V must be a basis (part (a) in Corollary 2 on page 47). Since $\{u, v, w\}$ spans V, it suffices to show that $u, v, w \in \operatorname{Span}(u + v + w, v + w, w)$. This is indeed the case: $w = 0(u + v + w) + 0(v + w) + 1(w)$, $v = 0(u + v + w) + 1(v + w) + (-1)w$ and $u = 1(u + v + w) + (-1)(v + w) + 0(w)$.

Problem 4. Is there a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$, so that

$$\begin{aligned} T(1,2) &= (3,4,5) \\ T(6,7) &= (8,9,10)? \end{aligned}$$

If yes, compute $T(11, 12)$. If not, explain why not.

Solution. We note that $(1, 2)$ and $(6, 7)$ are linearly independent. Indeed, if $\alpha(1, 2) + \beta(6, 7) = 0$, with α, β not both zero, this would imply that $(1, 2)$ is proportional to $(6, 7)$, which is not true. Thus $\{(1, 2), (6, 7)\}$ is a basis for $\mathbb{R}^2$. Now by Theorem 2.6 on p. 72, there exists a linear transformation T which maps $(1, 2)$ and $(6, 7)$ to any prescribed pair of vectors, e.g., $(3, 4, 5)$ and $(8, 9, 10)$.

To compute $T(11, 12)$, we must first express $(11, 12)$ in our basis. We have

$$(11, 12) = 2(6, 7) + (-1)(1, 2).$$

Thus

$$T(11, 12) = 2T(6, 7) - T(1, 2) = 2(8, 9, 10) - (3, 4, 5) = (13, 14, 15).$$

Problem 5. Let $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ be given by

$$T(A) = \operatorname{tr}(A).$$

(a) Find a basis for the null space of T.
(b) Find the dimensions of the null space and the range of T.
(c) Let α be the standard basis for $M_{2\times 2}(\mathbb{R})$ and let γ be the standard basis for $\mathbb{R}$. Find the matrix $[T]_\alpha^\gamma$.

Solution. (b) Note that the transformation T is clearly onto (For any $\lambda \in \mathbb{R}$ we have $\mathrm{tr}(\lambda I) = 2\lambda$, where I is the identity matrix). Thus the dimension of the range of T is 1. By the rank-nullity theorem, the null space of T has dimension $\dim(M_{2\times 2}) - 1 = 4 - 1 = 3$.

(a) Next, note that the matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

all belong to the null space of T, and clearly form a linearly independent set. Thus this set of three matrices is a basis for the null space of T.

(c) We compute:

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1,\ T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0,\ T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

Thus the matrix of T is

$$(1, 0, 0, 1).$$