Math 115a
Lecture 1
Fall 2009

Final exam

Name: ______________________________

Instructions:

- There are 10 problems. Make sure you are not missing any pages.
- Unless stated otherwise, you may use without proof anything proven in the sections of the book covered by this test.
- You may only cite an exercise from the book if it was assigned as homework.
- You must prove your answers (OF COURSE!).

Question	Points	Score
1	10	
2	15	
3	10	
4	15	
5	15	
6	10	
7	10	
8	15	
9	10	
10	15	
Total:	125	

1. (10 points) Let A be an upper triangular matrix. Use the definition of the determinant to prove that $\det(A)$ is the product of the diagonal entries of A.

Solution

We proceed by induction. The claim is easy to show for small matrices. Suppose it is true for upper triangular $(n-1)\times(n-1)$ matrices. Let

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ 0 & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{pmatrix}.$$

Recall that if A is $n \times n$, then

$$\det A = \sum_{i=1}^{n} (-1)^{1+i} a_{i1} \det(\widetilde{A_{i1}}),$$

where $\widetilde{A_{i1}}$ is the $(i,1)$-cofactor of A. Since A is upper triangular, $a_{i1} = 0$ if $i \geq 2$. Hence

$$\det A = \sum_{i=1}^{n} (-1)^{1+i} a_{i1} \det(\widetilde{A_{i1}}) = a_{11} \det(\widetilde{A_{11}}).$$

But $\widetilde{A_{11}}$ is an upper triangular $(n-1)\times(n-1)$ matrix, so the induction hypothesis applies, giving us $\det(\widetilde{A_{11}}) = a_{22}a_{33}\cdots a_{nn}$. Hence $\det A = a_{11}a_{22}a_{33}\cdots a_{nn}$.

2. (15 points) Let V be a finite dimensional inner product space. Let W be a subspace of V, and let $\{w_1, ..., w_k\}$ be an orthonormal basis of W. Define

$$T(x) = \sum_{j=1}^{k} \langle x, w_j \rangle w_j.$$

(a) (5 points) Prove $N(T) = W^\perp$
(b) (5 points) Prove $R(T) = W$.
(c) (5 points) Compute $T^*(x)$ for any x in V.

Solution (a)

Suppose $T(x) = 0$. Then $\sum_{j=1}^{k} \langle x, w_j \rangle w_j = 0$. Since $\{w_1, ..., w_k\}$ is orthonormal, it is linearly independent, which implies that $\langle x, w_j \rangle = 0$ for $j = 1, ...k$. This proves $N(T) \subseteq W^\perp$. On the other hand, if $x \in W^\perp$, then $\langle x, w_j \rangle = 0$ for $j = 1, ...k$, which implies that $T(x) = \sum_{j=1}^{k} \langle x, w_j \rangle w_j = 0$. This proves $W^\perp \subseteq N(T)$.

Solution (b)

Let $x \in W$. Say $x = \sum_{i=1}^{k} a_i w_i$. Then

$$\begin{aligned} T(x) &= \sum_{j=1}^{k} \langle \sum_{i=1}^{k} a_i w_i, w_j \rangle w_j \\ &= \sum_{j=1}^{k} \sum_{i=1}^{k} a_i \langle w_i, w_j \rangle \\ &= \sum_{i=1}^{k} a_i w_i \\ &= x, \end{aligned}$$

which implies that $W \subseteq R(T)$. On the other hand, if $y \in V$, then

$$T(y) = \sum_{j=1}^{k} \langle y, w_j \rangle w_j \in \text{span}\{w_1, ..., w_k\} \subseteq W,$$

which implies that $R(T) \subseteq W$.

Solution (c)

For any $x, y \in V$,

$$\begin{aligned} \langle T(x), y \rangle &= \langle \sum_{j=1}^{k} \langle x, w_j \rangle w_j, y \rangle \\ &= \sum_{j=1}^{k} \langle x, w_j \rangle \langle w_j, y \rangle \\ &= \sum_{j=1}^{k} \langle x, \overline{\langle w_j, y \rangle} w_j \rangle \\ &= \langle x, \sum_{j=1}^{k} \langle y, w_j \rangle w_j \rangle \\ &= \langle x, T(y) \rangle. \end{aligned}$$

Hence $T^* = T$.

3. (10 points) Let V be a finite dimensional inner product space. Let W be a subspace of V. Prove $(W^{\perp})^{\perp} = W$.

Solution

Let $w \in W$ and let $x \in W^{\perp}$. By definition of $W^{\perp}$, we have

$$\langle w, x \rangle = 0.$$

Since this holds for any $x \in W^{\perp}$, we have $w \in (W^{\perp})^{\perp}$. This proves $W \subseteq (W^{\perp})^{\perp}$.

Now let $y \in (W^{\perp})^{\perp}$. By a theorem in the book, there exists unique $w \in W$ and $z \in W^{\perp}$ such that $y = w + z$. Since we assumed $y \in (W^{\perp})^{\perp}$, we have $\langle y, z \rangle = 0$. Hence

$$0 = \langle y, z \rangle = \langle w + z, z \rangle = \langle w, z \rangle + \langle z, z \rangle = \langle z, z \rangle.$$

But this implies $z = 0$, which implies $y = w \in W$. This proves $(W^{\perp})^{\perp} \subseteq W$.

4. (15 points) Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Compute A^{100}.

Solution

An easy computation shows that the characteristic polynomial of A is $(\lambda - 3)(\lambda + 1)$, which means that the eigenvalues of A are 3 and -1. Another straightforward computation shows that the eigenspaces are

$$E_{-1} = \text{span}\{(1, -1)\}$$

and

$$E_3 = \text{span}\{(1, 1)\}.$$

Let $\beta = \{(1, -1), (1, 1)\}$. Then

$$Q = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

is the change of basis matrix from β to the standard ordered basis of $\mathbb{R}^2$. If we let

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix},$$

then

$$A = QDQ^{-1}.$$

Hence

$$\begin{aligned} A^{100} = \left(QDQ^{-1}\right)^{100} &= QD^{100}Q^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3^{100} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 3^{100} + 1 & 3^{100} - 1 \\ 3^{100} - 1 & 3^{100} + 1 \end{pmatrix} \end{aligned}$$

5. (15 points) Define $T\colon P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by

$$T(f) = \begin{pmatrix} f(0) & f(1) \\ f(1) & -f(0) \end{pmatrix}$$

(a) (5 points) Prove T is linear.
(b) (5 points) Compute $N(T)$.
(c) (5 points) Find a basis for $R(T)$.

Solution (a)

Let $\lambda \in \mathbb{R}$ and $f, g \in P_2(\mathbb{R})$. Then

$$\begin{aligned} T(f+\lambda g) &= \begin{pmatrix} (f+\lambda g)(0) & (f+\lambda g)(1) \\ (f+\lambda g)(1) & -(f+\lambda g)(0) \end{pmatrix} \\ &= \begin{pmatrix} f(0)+\lambda g(0) & f(1)+\lambda g(1) \\ f(1)+\lambda g(1) & -f(0)-\lambda g(0) \end{pmatrix} \\ &= \begin{pmatrix} f(0) & f(1) \\ f(1) & -f(0) \end{pmatrix} + \lambda \begin{pmatrix} g(0) & g(1) \\ g(1) & -g(0) \end{pmatrix} \\ &= T(f) + \lambda T(g). \end{aligned}$$

Solution (b)

Note that $T(f) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ if and only if $f(0) = f(0) = 1$. If $f(x) = ax^2+bx+c$, we have $f(0) = c = 0$, and $f(1) = a+b+c = 0$, which implies $a = -b$. So $N(T) = \text{span}\{x^2 - x\}$.

Solution (c)

One basis for $R(T)$ is

$$\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}.$$

6. (10 points) Recall that $C([-\pi,\pi])$ is the real vector space of continuous real-valued functions defined on the interval $[-\pi,\pi]$. For functions $f,g \in C([-\pi,\pi])$, define

$$\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx.$$

(a) (5 points) Prove that this defines an inner product on $C([-\pi,\pi])$.

(b) (5 points) Prove

$$\int_{-\pi}^{\pi} \sqrt{|\sin x \cos x|}dx \leq 4.$$

Solution (a)
You only need to check the four criteria in the definition of an inner product in Section 6.1. See example 3 on page 331 of the textbook.
Solution (b)
First, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int_{-\pi}^{\pi} \sqrt{|\sin x \cos x|}dx &= \int_{-\pi}^{\pi} \sqrt{|\sin x|}\sqrt{|\cos x|}dx \\ &\leq \sqrt{\int_{-\pi}^{\pi} |\sin x|dx}\sqrt{\int_{-\pi}^{\pi} |\cos x|dx}.\end{aligned}$$

Then by some calculus,

$$\int_{-\pi}^{\pi} |\sin x|dx = 2\int_{0}^{\pi} \sin x dx = 4.$$

Similarly,

$$\int_{-\pi}^{\pi} |\cos x|dx = 4.$$

Hence

$$\int_{-\pi}^{\pi} \sqrt{|\sin x \cos x|}dx \leq \sqrt{4}\sqrt{4} = 4.$$

7. (10 points) Define $T\colon P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by $T(f) = f' + f$. Let $\beta = \{1, x, x^2\}$. Compute $[T]_\beta^\beta$.

Solution

$$\begin{aligned} T(1) = 1 &= 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\ T(x) = 1 + x &= 1 \cdot 1 + 1 \cdot x + 0 \cdot x^2 \\ T(x^2) = 2x + x^2 &= 0 \cdot 1 + 2 \cdot x + 1 \cdot x^2. \end{aligned}$$

Hence

$$[T]_\beta^\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

8. (15 points) (a) (5 points)Determine whether the matrix A below is diagonalizable over $\mathbb{R}$. Justify your answer.

$$A = \begin{pmatrix} 5 & 1 & 5 \\ 1 & 3 & 1 \\ 5 & 1 & 5 \end{pmatrix}.$$

(b)(5 points) Determine whether the matrix B below is diagonalizable over $\mathbb{R}$. Justify your answer.

$$B = \begin{pmatrix} 5 & 1 & 5 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{pmatrix}.$$

(c)(5 points) Determine whether the matrix C below is diagonalizable over $\mathbb{C}$. Justify your answer.

$$C = \begin{pmatrix} 1 & i \\ 1 & 1 \end{pmatrix}.$$

Solution (a)

Since A is symmetric, there exists an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of A. This implies A is diagonalizable.

Solution (b)

The eigenvalues of this matrix are 3, with multiplicity 1, and 5, with multiplicity 2. But

$$B - 5I = \begin{pmatrix} 0 & 1 & 5 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

has nullity 1, which is not equal to the multiplicity of the eigenvalue 5. Hence B is not diagonalizable.

Solution (c)

Note that

$$CC^* = \begin{pmatrix} 1 & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1+i \\ 1-i & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 & i \\ 1 & 1 \end{pmatrix} = C^*C.$$

In other words, C is normal. Since $\mathbb{C}$ is the underlying field, C is diagonalizable.

9. (10 points) For $f, g \in P_2(\mathbb{R})$, define

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

You may assume this is an inner product on $P_2(\mathbb{R})$. Note that $P_1(\mathbb{R})$ is a subspace of $P_2(\mathbb{R})$, so the inner product defined above is also defined for functions in $P_1(\mathbb{R})$.
(a) (5 points) Find an orthogonal basis for $P_1(\mathbb{R})$.
(b) (5 points) Find an orthogonal basis for $P_2(\mathbb{R})$.

Solution (a)

We know the set $\{1, x\}$ is linearly independent in $P_1(\mathbb{R})$, so it is enough to carry out the Gram-Schmidt process to turn this into an orthogonal set. So let $w_1 = 1$ and $w_2 = x$. Then define $v_1 = w_1$ and

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle v_1}{||v_1||^2} = x - \int_0^1 x \cdot 1dx = x - \frac{1}{2}.$$

Since we obtained v_1 and v_2 from the Gram-Schmidt process, we know they are orthogonal. (It is also easy to check that v_1 and v_2 are orthogonal.) Since we started with a set that spans $P_1(\mathbb{R})$, we know $\{v_1, v_2\}$ spans $P_1(\mathbb{R})$.

Solution (b)

We know the set $\{1, x, x^2\}$ is linearly independent in $P_2(\mathbb{R})$, so it is enough to carry out the Gram-Schmidt process as in part (a), with $w_3 = x^2$. We can start with $v_1 = 1$ and $v_2 = x - \frac{1}{2}$. Then define

$$\begin{aligned} v_3 &= w_3 - \frac{\langle w_3, v_1 \rangle}{||v_1||^2} v_1 - \frac{\langle w_3, v_2 \rangle}{||v_2||^2} v_2 \\ &= x^2 - \frac{1}{3} - \frac{\int_0^1 x^2(x - \frac{1}{2})dx}{\frac{1}{12}}(x - \frac{1}{2}) \\ &= x^2 - x + \frac{1}{6}. \end{aligned}$$

Again, the set $\{v_1, v_2, v_3\}$ is orthogonal because we obtained it from the Gram-Schmidt process, and it spans all of $P_2(\mathbb{R})$ because we started with a basis of $P_2(\mathbb{R})$.

10. (15 points) Suppose V is a vector space, suppose $S\colon V \to V$ and $T\colon V \to V$ are linear, and suppose that $ST = TS$. Assume that v is an eigenvector of T with associated eigenvalue λ. Let E_λ be the λ-eigenspace of T and assume $\dim(E_\lambda) = 1$.
(a) (10 points) Prove v is an eigenvector of S.
(b) (5 points) Is λ necessarily an eigenvalue of S? Prove it is or find a counterexample.
Solution (a)
First note that $\lambda S(v) = ST(v) = TS(v) = T(S(v))$. This implies that $S(v) \in E_\lambda$. Since we assume that E_λ is one dimensional, and since we assume $v \in E_\lambda$, we know $E_\lambda = \operatorname{span}\{v\}$. Hence $S(v) \in \operatorname{span}\{v\}$, which means there exists $c \in F$ such that $S(v) = cv$. This is what we needed to prove.
Solution (b)
No. Counterexample: Let

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $AB = BA = A$. Further, 1 is an eigenvalue of B with a one-dimensional eigenspace. However, 1 is not an eigenvalue of A.

Extra Scratch Paper: