MATH 115A
PRACTICE FINAL EXAMINATION

March 10th, 2003

Problem 1. True or False. For each of the following statements, indicate if it is true or false. This problem will be graded as follows: you will receive n points for a correct answer, 0 points if there is no answer, and $-n$ points if the answer is wrong.

1. The set of polynomials of degree exactly 3 is not a vector space.
2. The set $W = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + a_2 + a_3 = 1\}$ is a subspace of $\mathbb{R}^3$.
3. A subset of a linearly dependent set is linearly dependent.
4. If $\dim(V) = n$, any generating set of V contains at least n vectors.
5. If a set of vectors S generates vectors space V, any vector in V can be written as a linear combination of vectors in S in a unique way.
6. A linear transformation $T : V \to V$ carries linearly independent subsets of V into linearly independent subsets of V.
7. The function $\det : M_{n\times n}(F) \to F$ which maps a matrix A to its determinant $\det(A)$ is linear.
8. Every square matrix is similar to a diagonal one.
9. A linear operator on an n-dimensional vector space that has less then n distinct eigenvalues can not be diagonalizable.
10. For any non-zero vector x in an inner-product space V its norm $||x|| > 0$.

Problem 2. Let V be the set of all pairs (x, y), where x is a real number and y is a positive real number. Define addition on V by

$$(x, y) + (x, x') = (x + x', y \cdot y')$$

and scalar multiplication by

$$c(x, y) = (cx, y^c) \qquad \text{for } c \in \mathbb{R}$$

Let $\overrightarrow{0} = (0, 1)$.

1. Show that V is a vector space with these operations.
2. Find the dimension of V.
3. Let n be the dimension of V which you found in part 2 of this problem. Construct an explicit isomorphism from V to $\mathbb{R}^n$.

Problem 3. Let W_1 and W_2 be subspaces of a vector space V. Prove that $V = W_1 \oplus W_2$ if and only if each vector x in V can be uniquely written in the form $x = x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$.

(Recall that a vector space V is called the *direct sum of* W_1 and W_2 if W_1 and W_2 are subspaces of V such that $W_1 \cap W_2 = \{\vec{0}\}$ and $V = W_1 + W_2$, where $W_1 + W_2 = \{w_1 + w_2,\ w_1 \in W_1,\ w_2 \in W_2\}$).

Problem 4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(a, b, c) = (a + b, b + c, 0)$.

1. Show that T is a linear transformation.
2. Find the null space and the range of T.
3. Find the nullity and rank of T and verify the dimension theorem.
4. Find the matrix of T in the basis $\beta = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$.

Problem 5. Compute the determinant and the trace of the following matrix:

$$\begin{pmatrix} 0 & 2 & 1 & 3 \\ 1 & 0 & -2 & 2 \\ 3 & -1 & 0 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix};$$

Is this matrix invertible? If yes, compute the inverse, if not, exaplain why not.

Problem 6. Prove that an upper-diagonal matrix is invertible iff all its diagonal entries are non-zero.

Problem 7. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(a_1, a_2, a_3) = (3a_1 - 2a_3, a_2, 3a_1 + 4a_2)$. Prove that T is an isomorphism and find T^{-1}.

Problem 8. Let A and B be invertible matrices. Prove that AB is invertible and that $(AB)^{-1} = B^{-1}A^{-1}$.

Problem 9. Test the following matrices for diagonalizability. If the matrix A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $D = Q^{-1}AQ$.

1. $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$;
2. $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$;

Problem 10. Suppose that $A \in M_{n\times n}(F)$ has exactly two distinct eigenvalues, λ_1 and λ_2, and that $\dim(E_{\lambda_1}) = n - 1$. Prove that A is diagonalizable.

Problem 11. Let $T : V \to V$ be a linear operator on an inner product space V. Suppose that $||T(x)|| = ||x||$ for all $x \in V$. Show that T is one-to-one.

Problem 12. Let $V = P(\mathbb{R})$ with the inner product

$$\langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)g(t)dt$$

Let $W = P_3(\mathbb{R})$ be a subspace of V. Use Gramm-Schmidt orthonormalization process to obtain an orthnormal basis of $P_3(\mathbb{R})$ from the standard basis $\{1, x, x^2, x^3\}$ for P_3.