MATH 115A
PRACTICE FINAL EXAMINATION

Problem 1. True or False. For each of the following statements, indicate if it is true or false. This problem will be graded as follows: you will receive n points for a correct answer, 0 points if there is no answer, and $-n$ points if the answer is wrong.

1. The set of polynomials of degree exactly 3 is not a vector space.
   FALSE: for example, the sum of $x^3 + 1$ and $-x^3$ is not of degree 3. Hence, this space is not closed under addition.
2. The set $W = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + a_2 + a_3 = 1\}$ is a subspace of $\mathbb{R}^3$.
   FALSE: notice that the zero vector is not in W.

3. A subset of a linearly dependent set is linearly dependent.
   FALSE: for a non-zero vector $v \in V$ and a constan $a \in F$, the set $\{v, a \cdot v\}$ is linearly dependent. However, its subset consisting of just v is linearly independent.
4. If $\dim(V) = n$, any generating set of V contains at least n vectors.
   TRUE: since $\dim(V) = n$, the number of elements in a basis is n. The number of elements in a generating set is bigger or equal then the number of elements in a basis.

5. If a set of vectors S generates vectors space V, any vector in V can be written as a linear combination of vectors in S in a unique way.
   FALSE: for example, vectors $(1,0)$, $(1,1)$ and $(0,1)$ generate $\mathbb{R}^2$. However, $(2,2)$ can be written as a linear combination of these vectors in more then one way: e.g., $(2,2) = 2 \cdot (1,1) = 2 \cdot (1,0) + 2 \cdot (0,1)$.

6. A linear transformation $T : V \to V$ carries linearly independent subsets of V into linearly independent subsets of V.
   FALSE: for a set oflinearly independent vectors in V we can construct a linear transformation which maps this set into any (not necessarily linearly independent) given set of V. For example, the zero transformation maps all the vectors into zero, which forms a linearly dependent set.

7. The function $\det : M_{n\times n}(F) \to F$ which maps a matrix A to its determinant $\det(A)$ is linear.
   FALSE: The determinant has the property of n-linearity, but not the usual linearity.

8. Every square matrix is similar to a diagonal one.
   FALSE: If a matrix is similar to a diagonal, it is diagonalizable. Not all matrices are. For example, $A = ...$ is not.

9. A linear operator on an n-dimensional vector space that has less then n distinct eigenvalues can not be diagonalizable.
   FALSE: The identity operator on an n-dimensional vector space has exactly one distinct eigenvalue ($\lambda = 1$), but is clearly diagonalizable.

10. For any non-zero vector x in an inner-product space V its norm $||x|| > 0$.
    TRUE: it is one of the axioms of an inner product.

Problem 2. Let V be the set of all pairs (x, y), where x is a real number and y is a positive real number. Define addition on V by

$$(x, y) + (x, x') = (x + x', y \cdot y')$$

and scalar multiplication by

$$c(x, y) = (cx, y^c) \qquad \text{for } c \in \mathbb{R}$$

Let $\overrightarrow{0} = (0, 1)$.

1. Show that V is a vector space with these operations.
2. Find the dimension of V.
3. Let n be the dimension of V which you found in part 2 of this problem. Construct an explicit isomorphism from V to $\mathbb{R}^n$.

Solution. (1) We must check the axioms of a vector space. Commutativity and associativity of addition follow from the respective properties of addition and multiplication of numbers. Next,

$$\overrightarrow{0} + (x, y) = (0 + x, 1 \cdot y) = (x, y).$$

Given (x, y), we have

$$(x, y) + (-x, y^{-1}) = (x - x, yy^{-1}) = \overrightarrow{0}.$$

Thus inverses exist and $-(x, y) = (-x, y^{-1})$.

We have

$$1 \cdot (x, y) = (1 \cdot x, y^1) = (x, y).$$

For a, b scalars,

$$(ab)(x, y) = (abx, y^{ab}) = (a(bx), (y^b)^a) = a(bx, y^b) = a(b(x, y)).$$

Also,

$$a((x, y) + (x', y')) = a(x + x', yy') = (ax + ax', (yy')^a) = a(x, y) + a(x', y').$$

Lastly,

$$(a + b)(x, y) = ((a + b)x, y^{a+b}) = (ax + bx, y^a y^b) = a(x, y) + b(x, y).$$

(2) We claim that the set $\{v, w\}$, where $v = (1, 0)$ and $w = (0, e)$ is a basis for V (here e is the base of the natural logarithm, $e = 2.718281828459045...$) First, the two vectors are linearly independent. If $av + bw = 0$, then $a(1, 0) + b(0, e) = \overrightarrow{0} = (0, 1)$. Thus $a = 0$ and $e^b = 1$, so that $b = 0$. Next, any vector (x, y) can be written as $(x, y) = (x \cdot 1, e^{\log y}) = x(1, 0) + y(0, e) = xv + \log yw$. Thus V is the span of $\{v, w\}$. So $\{v, w\}$ is a basis and thus $\dim V = 2$.

(3) We take the unique linear map from V to $\mathbb{R}^2$, which takes v to $(1, 0)$ and w to $(0, 1)$. Explicitely, the map is

$$T(x, y) = T(xv + \log yw) = (x, \log y).$$

It is easily checked that T is linear. Since it maps a basis to a basis, it is an isomorphism. The inverse of T is $(x, y) \mapsto (x, e^y)$.

Problem 3. Let W_1 and W_2 be subspaces of a vector space V. Prove that $V = W_1 \oplus W_2$ if and only if each vector x in V can be uniquely written in the form $x = x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$.

(Recall that a vector space V is called the *direct sum of* W_1 and W_2 if W_1 and W_2 are subspaces of V such that $W_1 \cap W_2 = \{\overrightarrow{0}\}$ and $V = W_1 + W_2$, where $W_1 + W_2 = \{w_1 + w_2,\ w_1 \in W_1,\ w_2 \in W_2\}$).

Solution. Assume first that $V = W_1 \oplus W_2$. Let $x \in V$. Since by assumption $V = W_1 + W_2$, it follows that there exist $x_1 \in W_1$ and $x_2 \in W_2$ so that $x = x_1 + x_2$. We claim that x_1 and x_2 with this property are unique. Assume that $x = x_1' + x_2'$ with $x_1' \in W_1$ and $x_2' \in W_2$. Then

$$0 = x - x = (x_1 - x_1') + (x_2 - x_2').$$

It follows that

$$x_1 - x_1' = x_2' - x_2.$$

Since $x_1, x_1' \in W_1$, $x_1 - x_1' \in W_1$. Since $x_2, x_2' \in W_2$, $x_2' - x_2 \in W_2$. Thus $x_1 - x_1' = x_2' - x_2$ lies in both W_1 and W_2 and thus is in the intersection $W_1 \cap W_2$. But $W_1 \cap W_2$ was assumed to consist of the zero vector. So $x_1 - x_1' = x_2' - x_2 = \overrightarrow{0}$. But then $x_1 = x_1'$ and $x_2 = x_2'$. For the converse, assume that every $x \in W$ can be written uniquely as $x = x_1 + x_2$ with $x_1 \in W_1$ and $x_2 \in W_2$. Then clearly $W = W_1 + W_2$. So we need to check that $W_1 \cap W_2 = \{\overrightarrow{0}\}$. Let $x \in W_1 \cap W_2$. Then

$$\begin{aligned} x &= x_1 + x_2, \qquad x_1 = \overrightarrow{0} \in W_1, x_2 = x \in W_2 \\ x &= x_1' + x_2', \qquad x_1' = x \in W_1, x_2' = \overrightarrow{0} \in W_2 \end{aligned}$$

are two ways of writing x as a sum of vectors from W_1 and W_2 . Since by assumption the way to represent x as such a sum is unique, it follows that $x_1 = x_1'$, so that $x = \overrightarrow{0}$. Thus $W_1 \cap W_2 = \{\overrightarrow{0}\}$ and so $V = W_1 \oplus W_2$.

Problem 4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(a, b, c) = (a + b, b + c, 0)$.

1. Show that T is a linear transformation.
2. Find the null space and the range of T.
3. Find the nullity and rank of T and verify the dimension theorem.
4. Find the matrix of T in the basis $\beta = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$.public_html/115a.1.03w/27p188.pdf

SOLUTION. 1. We must check that T preserves sums and scalar multiples. Let $\alpha \in \mathbb{R}$, and let $v, w \in \mathbb{R}^3$. Assume taht $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$. Then $\alpha v = (\alpha v_1, \alpha v_2, \alpha v_3)$ and $v + w = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$. We now compare:

$$\begin{aligned} T(v + w) &= (v_1 + w_1 + v_2 + w_2, v_2 + w_2 + v_3 + w_3, 0) \\ T(v) + T(w) &= (v_1 + v_2, v_2 + v_3, 0) + (w_1 + w_2, w_2 + w_3, 0) \quad = \quad T(v + w). \end{aligned}$$

Similarly,

$$\begin{aligned} T(\alpha v) &= (\alpha v_1 + \alpha v_2, \alpha v_2 + \alpha v_3, 0) \\ \alpha T(v) &= \alpha(v_1 + v_2, v_2 + v_3, 0) \quad = \quad T(\alpha v). \end{aligned}$$

Thus T is indeed linear.

2. It is clear that any vector in the range of T must have the third coordinate equal to zero. On the other hand, the vector $(a, b, 0) = T(a, 0, b)$. Thus the range of T consists of all vectors of the form $(a, b, 0)$ with a and b arbitrary numbers.

To find the null space of T , assume that $Tv = 0$ for some $v = (v_1, v_2, v_3)$. Thus $v_1 + v_2 = 0$ and $v_2 + v_3 = 0$. It follows that $v_2 = -v_1$ and $v_3 = -v_2$. Hence the null space of T consists of all vectors of the form $(a, -a, a)$, where a is an arbitrary number.

3. It follows from 2. that the null space is spanned by the vector $(1, -1, 1)$ and the range is spanned by the vectors $(1, 0, 0)$ and $(0, 1, 0)$, which are clearly linearly independent. Thus the nullity of T is one, and the rank is 2. The dimension theorem predicts that $2 + 1$ is the dimension of the vector space $\mathbb{R}^3$, which is indeed $2 + 1 = 3$.

4. We have: $T(1, 0, 0) = (1, 0, 0)$; $T(1, 1, 0) = (2, 1, 0) = (1, 0, 0) + (1, 1, 0)$; $T(1, 1, 1) = (2, 2, 0) = 2(1, 1, 0)$. Thus the matrix of T is

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

in the given basis.

Problem 5. Compute the determinant and the trace of the following matrix:

$$\begin{pmatrix} 0 & 2 & 1 & 3 \\ 1 & 0 & -2 & 2 \\ 3 & -1 & 0 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix};$$

Is this matrix invertible? If yes, compute the inverse, if not, exaplain why not.
ANSWERS. The trace is the sum of the diagonal entires and is zero. The determinant is -3.

Problem 6. Prove that an upper-diagonal matrix is invertible iff all its diagonal entries are non-zero.
PROOF: Let A be an upper-triangular matrix. Its determinant is equal to the product of the diagonal entries:

$$\det(A) = A_{11} \cdot A_{22} \cdot \dots \cdot A_{nn}$$

(This can be shown, for example, by induction on the size of the matrix. In the induction step, use expansion of the determinant in the first column).
Since a matrix is invertible iff its determinant is not zero, it follows that an A is invertible iff all its diagonal entries are not zero.

Problem 7. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(a_1, a_2, a_3) = (3a_1 - 2a_3, a_2, 3a_1 + 4a_2)$. Prove that T is an isomorphism and find T^{-1}.

PROOF: Relative to the standard basis, the matrix of T is

$$\begin{pmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ 3 & 4 & 0 \end{pmatrix}.$$

It is sufficient to prove that this matrix is invertible. Its determinant is, using the column expansion for the last column, $-2 \cdot (0 \cdot 4 - 1 \cdot 3) = 6 \neq 0$. Thus the matrix is invertble. The inverse of the matrix is

$$\begin{pmatrix} 0 & -\frac{4}{3} & \frac{1}{3} \\ 0 & 1 & 0 \\ -\frac{1}{2} & -2 & \frac{1}{2} \end{pmatrix},$$

so that T^{-1} is given by:

$$T^{-1}(a_1, a_2, a_3) = (-\frac{4}{3}a_2 - \frac{1}{3}a_3, a_2, -\frac{1}{2}a_1 - 2a_2 + \frac{1}{2}a_3).$$

Problem 8. Let A and B be invertible matrices. Prove that AB is invertible and that $(AB)^{-1} = B^{-1}A^{-1}$.
SOLUTION. We know that $\det(AB) = \det(A)\det(B)$. Since A and B are invertible, their determinants are nonzero. Thus $\det A \det B$ is also nonzero, so AB has a nonzero determinant, and so is invertible.
Since the inverse of a matrix is unique, we need only to check that $(AB) \cdot (B^{-1}A^{-1})$ is the identity matrix. Since matrix multiplication is associative, this is the same as $A(BB^{-1})A^{-1} = AA^{-1} = I$.

Problem 9. Test the following matrices for diagonalizability. If the matrix A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $D = Q^{-1}AQ$.

1. $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$;
2. $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$;

SOLUTION. 1. Consider the characteristic polynomial of A,

$$\begin{aligned} f(\lambda) = \det \begin{pmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ -1 & -1 & 1-\lambda \end{pmatrix} \\ = (3-\lambda)((4-\lambda)(1-\lambda)+2) - (2(2-\lambda)+2) + (-2+(4-\lambda)) \\ = \lambda^3 - 8\lambda^2 + 20\lambda - 16. \end{aligned}$$

Note that $\lambda = 2$ is clearly a root, since $8 - 32 + 40 - 16 = 0$. Dividing by $\lambda - 2$ gives $x^2 - 6x + 8$. This has two roots, 2 and 4. Thus the eigenvalues of A are $\{2, 4\}$, with the eigenvalue 2 having multiplicity 2. We try to find eigenvectors of A with eigenvalue 2:

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix} v = 2v.$$

We are interested in the dimension of the set of solutions of this equation, i.e., in the solutions to the equation

$$\begin{pmatrix} 3-2 & 1 & 1 \\ 2 & 4-2 & 2 \\ -1 & -1 & 1-2 \end{pmatrix} v = 0.$$

The following vectors solve this equation, and are therefore eigenvectors for A with eigenvalue 2:

$$v_1 = (0, -1, 1), \qquad v_2 = (2, -1, -1).$$

Similarly, the vector

$$v_3 = (-1, -2, 1)$$

is an eigenvector with eigenvalue 4. Thus the matrix is diagonalizable. Letting

$$Q = \begin{pmatrix} 0 & 2 & -1 \\ -1 & -1 & -2 \\ 1 & -1 & 1 \end{pmatrix},$$

we find that

$$Q^{-1} \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix} Q = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

as required.
2. The characteristic polynomial of A is $(1-\lambda)^2$ and so 1 is the only eigenvalue. Solving the equation

$$Av = v$$

leads us to finding all solutions to

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} v = 0.$$

The set of solution is one-dimensional and is spanned by $v = (1, 0)$, which is the sole eigenvector of A. It follows that A is not diagonalizable.

Problem 10. Suppose that $A \in M_{n\times n}(F)$ has exactly two distinct eigenvalues, λ_1 and λ_2, and that $\dim(E_{\lambda_1}) = n-1$. Prove that A is diagonalizable.

SOLUTION. Since $\dim E_{\lambda_2} \geq 1$ and $\dim E_{\lambda_1} + \dim E_{\lambda_2} = n - 1 + \dim E_{\lambda_2} \leq \dim \mathrm{Span}(E_{\lambda_1}, E_{\lambda_2}) \leq \dim F^n = n$, we find that $\dim E_{\lambda_2} = 1$. The multiplicity of λ_1 must be at most $n-1$, since the total degree of the characteristic polynomial is n, and it has another root, λ_2. On the other hand, the multiplicity of any eigenvalue is at least as large as the dimension of the associated eigenspace. Thus the multiplicity of λ_1 must be $n-1$, and the multiplicity of λ_2 is 1. Since the multiplicity of each eigenvalue is equal to the dimension of the corresponding eigenspace, the matrix A is diagonalizable.

Problem 11. Let $T : V \to V$ be a linear operator on an inner product space V. Suppose that $||T(x)|| = ||x||$ for all $x \in V$. Show that T is one-to-one.
SOLUTION. It suffices to show that $\ker T$ is zero. Let $x \in \ker T$. Then $T(x) = 0$ and so $\|T(x)\| = 0$. But we are given that $\|T(x)\| = \|x\|$, so that $\|x\| = 0$. Thus $x = 0$ and so $\ker T = \{0\}$.

Problem 12. Let $V = P(\mathbb{R})$ with the inner product

$$\langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)g(t)dt$$

Let $W = P_3(\mathbb{R})$ be a subspace of V. Use Gramm-Schmidt orthonormalization process to obtain an orthnormal basis of $P_3(\mathbb{R})$ from the standard basis $1, x, x^2, x^3$ for P_3.
SOLUTION. Let $v_1' = 1$. Then $\|v_1'\| = \sqrt{\langle 1, 1\rangle} = \sqrt{2}$. Let

$$v_1 = \frac{1}{\sqrt{2}}.$$

To find v_2, we set $v_2'' = x$. Then $v_2' = v_2'' - \langle v_2'', v_1\rangle v_1$ is perpendicular to v_1. Since $\langle v_2'', v_1\rangle = 0$ (by symmetry), we find that $v_2' = x$. We have $\|v_2'\| = \sqrt{\langle x, x\rangle} = \sqrt{\frac{2}{3}}$. Let

$$v_2 = \sqrt{\frac{3}{2}}x.$$

Then $\{v_1, v_2\}$ is an orthonormal set. To find v_3, we set $v_3'' = x^2$. Then $v_3' = v_3'' - \langle v_3'', v_2\rangle v_2 - \langle v_3'', v_1\rangle v_1$ is perpendicular to both v_1 and v_2. We have $\langle v_3'', v_2\rangle = 0$ by symmetry and

$$\langle v_3'', v_1\rangle = \frac{1}{\sqrt{2}}\int_{-1}^{1} x^2 dx = \frac{2}{3\sqrt{2}} = \frac{\sqrt{2}}{3}.$$

Thus

$$v_3' = x^2 - \frac{\sqrt{2}}{3}v_1 = x^2 - \frac{1}{3}.$$

Then

$$\|v_3'\| = \sqrt{\langle v_3', v_3'\rangle} = \sqrt{\frac{8}{45}} = \frac{2\sqrt{2}}{3\sqrt{5}}.$$

Thus if we set

$$v_3 = \sqrt{\frac{45}{8}}x^2 - \frac{\sqrt{5}}{2\sqrt{2}},$$

then $\{v_1, v_2, v_3\}$ is an orthonormal set. Finally, we let $v_4'' = x^3$. Then $v_4' = v_4'' - \langle v_4'', v_3\rangle v_3 - \langle v_4'', v_2\rangle v_2 - \langle v_4'', v_1\rangle$ is orthogonal to $\{v_1, v_2, v_3\}$. Because of symmetry, v_4 is orthogonal to any even polynomial, so only $\langle v_4'', v_2\rangle$ can be nonzero. We have

$$\langle v_4'', v_2\rangle = \sqrt{\frac{3}{2}}\int_{-1}^{1} x^4 dx = \sqrt{\frac{3}{2}} \cdot \frac{2}{5} = \frac{\sqrt{6}}{5}.$$

Thus $v_4' = x^3 - \frac{\sqrt{6}}{5}\sqrt{\frac{3}{2}}x = x^3 - \frac{3}{5}x$. Now, $\|v_4'\| = \sqrt{\frac{8}{175}}$.Thus if we set

$$v_4 = \frac{5}{2}\sqrt{\frac{7}{2}}x^3 - \frac{3}{2}\sqrt{\frac{7}{2}}x,$$

then $\{v_1, v_2, v_3, v_4\}$ is the desired orthonormal basis.v8