1 Problem 1.6.7

The vectors $u_1 = (2, -3, 1)$, $u_2 = (1, 4, -2)$, $u_3 = (-8, 12, -4)$, $u_4 = (1, 37, -17)$, and $u_5 = (-3, -5, 8)$ generate $\mathbb{R}^3$. Find a subset of the set $\{u_1, u_2, u_3, u_4, u_5\}$ that is a basis for $\mathbb{R}^3$.

Solution: To do so, it suffices to find a linearly independent subset. This is easily done by picking u_1 and u_2, which are clearly independent, and verifying independence with the other vectors. Doing so gives that u_5 is independent from u_1 and u_2:

$$\begin{bmatrix} 2 & 1 & -3 \\ -3 & 4 & -5 \\ 1 & -2 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 1/2 & -3/2 \\ 0 & 1 & -1/11 \\ 1 & -2 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 1/2 & 3/2 \\ 0 & 1 & -1/11 \\ 0 & -5/2 & 13/2 \end{bmatrix}$$

which reduces, of course, to,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is sufficient to show independence. Thus, the subset $\{u_1, u_2, u_5\}$ is a basis for $\mathbb{R}^3$.

2 Problem 1.6.13

The set of solutions to the system,

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_1 - 3x_2 + x_3 = 0$$

Solution: Well, first, we notice that if we add the second equation's negation to the first, we have,

$$-x_1 + x_2 = 0$$

in other words, the system is satisfied for x_1, x_2 so that $x_1 = x_2$. x_3 is seen to depend on the choice of x_1, since plugging in x_1 for x_2 gives that the system is satisfied for $x_1 = x_3$. What emerges is that the solution set of this system is,

$$S = \text{span}(1, 1, 1)$$

which is, in fact, a subspace of $\mathbb{R}^3$.

3 Problem 1.6.19

Complete the proof of Theorem 1.8.

Proof: It remains to be seen that (using the same notation as in the text), if each $v \in V$ can be uniquely represented as a linear combination of vectors of β, then β is a basis of V. Suppose the vectors in β are not linearly independent. Then there exist scalars α_i not all zero such that,

$$\sum_{i=1}^{n} \alpha_i \beta_i = 0$$

But notice that $\alpha_i = 0$ for all i also solves the equation. Thus, 0 has at least two different linear representations, which contradicts uniqueness. Thus, β is linearly independent. Now, we need to show that $V = \text{span}() = \beta)$. If $v \in V$, then v can be represented as a linear combination of elements in β and thus $v \in \text{span}(\beta)$. If $v \in \text{span}(\beta)$, then obviously $v \in V$ by closure of V under addition and scalar multiplication. Thus, $v = \text{span}(\beta)$ and β is a basis for V.

4 Problem 2.1.2

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$. Prove that T is linear and find bases for $N(T)$ and $R(T)$. Then, compute the nullity and rank of T, and verify the dimension theorem. Finally, use the appropriate theorems to determine whether T is injective or surjective.

Solution: Let's start with linearity. We take two vectors, (a, b, c) and (x, y, z) in $\mathbb{R}^3$. Then $T(a+x, b+y, c+z) = (a+x-b-y, 2c+2z)$ by definition. Then, take $T(a, b, c) + T(x, y, z)$. This is just $(a-b, 2c) + (x-y, 2z) = (a+x-b-y, 2c+2z)$. Finally, take $T(ka, kb, kc)$ for $k \in F$. This is $(ka - kb, 2kc)$. Also, $kT(a, b, c) = k(a-b, 2c) = (ka - kb, 2kc)$. We conclude that T is linear by definition.

Now, for $N(T)$, suppose $T(a, b, c) = 0$. Then $(a - b, 2c) = 0$. Thus, for any vector (a, b, c) such that $a = b$ and $c = 0$, we have that $(a, b, c) \in N(T)$. Hence, the basis is $N(T) = \text{span}(1, 1, 0)$. Now, we claim that any vector in $\mathbb{R}^2$ can be written using transformed elements of $\mathbb{R}^3$. Well, take $(x, y) \in \mathbb{R}^2$. Then we want to see if,

$$(x, y) = (a - b, 2c)$$

for some real a, b, c. But the equation $a - b = x$ has infinitely many solutions. Finally, if $\frac{1}{2}y = c$, then we have the desired result. Since the range is all of $\mathbb{R}^2$, we can simply use the standard basis as a basis for $R(T)$.

Finally, we notice that T is not injective since it's nullspace does not consist only of the zero vector. However, by the previous argument, the transformation is onto (since its range is all of $\mathbb{R}^2$).

5 Problem 2.1.15

Recall the definition of $P(\mathbb{R})$ given on page 10 of the text. Define $T : P(\mathbb{R}) \to P(\mathbb{R})$ by,

$$T(f(x)) = \int_0^x f(t)dt$$

Prove that T is linear and injective, but not surjective.

Proof: Linearity is straightforward. Take $f, g \in P(\mathbb{R})$. Then,

$$T(f+g) = \int_0^x (f+g)dt = \int_0^x f dt + \int_0^x g dt = T(f) + T(g)$$

In the most pure sense, however, this is not completely justified. To justify this completely, you need at least one academic quarter of real analysis and a thorough understanding of *partitions* and *integrability* (it suffices, in this case, to just understand the Riemann Integral; yes, there are other types of integrals!). For now, just rely on the properties you learned in elementary calculus.

Now, we show injectivity. Suppose $T(f) = T(g)$ for some f, g. Then,

$$\int_0^x f dt = \int_0^x g dt$$

Now, we have the following,

$$\int_0^x f dt - \int_0^x g dt = 0$$

By the fundamental theorem of calculus (which you can assume), we differentiate both sides

$$f(x) - g(x) = 0$$

Thus, since $f(x) = g(x)$, we have proven injectivity. Another way of doing this is by showing that the nullspace is zero. This is equally valid:

$$\int_0^x f(t)dt = 0$$

differentiating,

$$f(x) = 0$$

for arbitrary $f(x)$. Thus, $N(T) = \{0\}$; that is, the zero function. However, it is not onto. To see this, notice that for any constant function c, there exists no function in $P(\mathbb{R})$ so that $P(f) = c$.

6 Problem 2.1.18

Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $N(T) = R(T)$.

Solution: An example of such is this. Let,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

That is, for any (x, y), $T(x, y) = (y, 0)$. Then we can see that $N(T) = \text{span}(1, 0)$ and so is $R(T)$.

7 Online Problem 1

Let $T : V \to W$ be a linear map, show the null space, null(T), is a subspace of V.

Proof: Choose $x, y \in (T)$. Choose $c \in F$. Notice that $0 = T(x) + T(y) = T(x + y)$. Hence, $x + y \in \text{null}(T)$. But also, $0 = cT(x) = T(cx)$. Thus, $cx \in \text{null}(T)$. We conclude that null(T) is a subspace by definition.

8 Problem 2.2.4

Define $T : M_{2\times2}(\mathbb{R}) \to P_2(\mathbb{R})$ by,

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a + b) + (2d)x + bx^2$$

Let,

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

and $\gamma = \left\{1, x, x^2\right\}$. Compute $[T]_\beta^\gamma$.

Solution: Apply T to the basis elements to get 1, $1 + x^2$, 0, and $2x$ respectively. Then, if β_i represent the various elements of β, we take,

$$[T]_\beta^\gamma = \left([T(\beta_1)]_\gamma, [T(\beta_2)]_\gamma, [T(\beta_3)]_\gamma, [T(\beta_4)]_\gamma \right)$$

This is,

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

9 Problem 2.2.5a

Let,

$$\alpha = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

and $\beta = \left\{1, x, x^2\right\}$. Finally, let $\gamma = \{1\}$. Define $T : M_{2\times2}(F) \to M_2 \times 2(F)$ by $T(A) = A^t$. Compute $[T]_\alpha$.

Solution: This is straightforward. All we must do, again, is apply the transformation to the basis elements. Let α_i denote the basis elements of α. Then $T(\alpha_1)$ and $T(\alpha_4)$ remain unchanged. However, $T(\alpha_2) = \alpha_3$. Also, $T(\alpha_3) = T(\alpha_2)$. The matrix, then, is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

10 Problem 2.2.16

Let V and W be vector spaces such that $\dim(V) = \dim(W)$, and $T : V \to W$ be linear. Show that there exist ordered bases β and γ for V and W respectively, such that $[T]_\beta^\gamma$ is a diagonal matrix.

Proof: Let $\alpha = \{v_1, ..., v_k\}$ be an ordered basis for $N(T)$. Extend α to an ordered basis for the whole space: $\beta = \{v_1, ..., v_k, v_{k+1}, ..., v_n\}$. Now, we write the vectors $\{T(v_{k+1}), ..., T(v_n)\}$ as are an ordered basis for $R(T)$ (as per the proof of the dimension theorem). Extend this to the ordered basis $\gamma = \{w_1, ..., w_k, T(v_{k+1}), ..., T(v_n)\}$ for W. Now,

$$[T(v_i)]_\gamma = \begin{cases} 0 & \text{if } 1 \leq i \leq k \\ e_i & \text{if } k+1 \leq i \leq n \end{cases}$$

This gives the desired result,

$$[T]_\beta^\gamma = (0, ..., 0, e_{k+1}, ..., e_k)$$

That is,

$$\begin{bmatrix} 0 & 0 \\ 0 & \mathrm{I}_{\frac{n}{2}\times\frac{n}{2}} \end{bmatrix}$$