1 Problem 1.2.7

In any vector space V, show that $(a+b)(x+y) = ax + ay + bx + by$ for any $x, y \in V$ and any $a, b \in F$.

Proof: By definition, F is closed under addition, so write $a + b := c \in F$. Then we have,

$$c(x+y) = cx + cy$$

by VS7. But now,

$$(a+b)x + (a+b)y = ax + bx + ay + by$$

by VS8. Finally, this is $ax + ay + bx + by$ by VS1.

2 Problem 1.2.13

Let V denote a set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in \mathbb{R}$, define,

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$$

and,

$$c(a_1, a_2) = (ca_1, a_2)$$

Is V a vector space over $\mathbb{R}$ under these operations? Justify your answer.

Solution: Nope. Check VS8 ($c, d \in \mathbb{R}$):

$$(c+d)(a_1, a_2) = (ca_1 + da_1, a_2)$$

But notice,

$$c(a_1, a_2) + d(a_1, a_2) = (ca_1 + da_2, a_2^2)$$

by definition. Hence, VS8 fails.

3 Problem 1.2.17

Let $V = \{(a_1, a_2) : a_1, a_2 \in F\}$ where F is a field. Define addition of elements of V coordinatewise and, for $c \in F$ and $(a_1, a_2) \in V$, define,

$$c(a_1, a_2) = (a_1, 0)$$

Is V a vector space over F with these operations? Justify your answer.

Solution: Again, no is the answer. There doesn't exist an identify scalar element! Choose a scalar c. Then,

$$c(a_1, a_2) = (a_1, 0) = (a_1, a_2)$$

only if $a_2 = 0$. Thus, not every element $x \in V$ has the property that the scalar identify (whatever that may be) takes it to itself.

4 Problem 1.3.4

Prove that $(A^t)^t = A$ for any $A \in M_{m\times n}(F)$.

Proof: Choose an entry of A and call it A_{ij} where $1 \leq i \leq m$ and $1 \leq j \leq n$. Take A^t. Then by definition, the A^t_{ji} element of A^t is A_{ij}. Take $(A^t)^t$. Then the $(A^t)^t_{ij}$ element of $(A^t)^t$ is the A^t_{ji} element from A^t. But this is exactly A_{ij}. Thus, the $(A^t)^t_{ij}$ element is A_{ij}. Since A_{ij} is arbitrary, we conclude that $A = (A^t)^t$.

5 Problem 1.3.19

Let W_1, W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace if and only if $W_1 \subseteq W_2$ or $W_1 \supseteq W_2$.

Proof: Well, to prove this statement, first show that, if W_1, W_2 are subspaces of a vector space V and $W_1 \cup W_2$ is a subspace, then $W_1 \subseteq W_2$ or $W_1 \supseteq W_2$. Choose $v_1, v_2 \in W_1, W_2$ respectively. Then obviously $v_1, v_2 \in W_1 \cup W_2$. But by our assumption, $v_1 + v_2 \in W_1 \cup W_2$. This indicates that $v_1 + v_2 \in W_1$ or $v_1 + v_2 \in W_2$.

Now, suppose the former. Then again, by closure under addition, $v_1 + v_2 - v_1 \in W_1$. Thus, $v_2 \in W_1$. Since we chose $v_2 \in W_2$, we have shown than $W_1 \supseteq W_2$. Now, suppose that $v_1 + v_2 \in W_2$. Then by the same reasoning, we have that $v_1 + v_2 - v_2 \in W_2$. Thus, $v_1 \in W_2$. But v_1 was chosen arbitrarily in w_1. Thus, in this case, $W_1 \subseteq W_2$. It follows that either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Now, we must prove the converse. We wish to show that, given either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$, we have that $W_1 \cup W_2$ is a subspace of V. Choose a scalar $c \in F$ and a vector $v \in W_1 \cup W_2$. Then v is an element of W_1 or W_2. But in either case, since W_1 and W_2 are subspaces, cv is an element of W_1 or W_2. Thus, $cv \in W_1 \cup W_2$. Now, choose $v_1, v_2 \in W_1 \cup W_2$. If both vectors are in one subspace, the conclusion follows (why?). Suppose then, without loss of generality, that $v_1 \in W_1$ and $v_2 \in W_2$.

Suppose also that $W_1 \subseteq W_2$. Then since $v_1 \in W_1$ implies that $v_1 \in W_2$ and W_2 is a subspace, then $v_1 + v_2 \in W_2$ and, thus, $v_1 + v_2 \in W_1 \cup W_2$. A similar result holds for $W_1 \supseteq W_2$ (see if you can show this).

Finally, it remains to be seen that the zero vector is in $W_1 \cup W_2$. But this is clear, since the zero vector is in W_1 and W_2. It must also, therefore, be in their union.

We conclude that $W_1 \cup W_2$ is a subspace and the proof is complete.

6 Problem 1.3.20

Prove that if W is a subspace of a vector space V and $w_1, ..., w_n$ are in W for arbitrary n, then,

$$\sum_{i=1}^{n} a_i w_i \in W \tag{1}$$

for any $a_1, ..., a_n \in F$ (the field).

Proof: If $n = 1$, then we simply have that $w_1 \in W$. Since W is a subspace, then by definition, $a_1 w_1 \in W$. Suppose then that (1) holds for n and consider $n + 1$. Then we want to show that,

$$\sum_{i=1}^{n} a_i w_i + a_{n+1} w_{n+1} \in W \tag{2}$$

But by assumption, $\sum a_i w_i \in W$. Also, by similar reasoning to above, since W is a subspace, $a_{n+1} w_{n+1} \in W$. Finally then, let $\sum a_i w_i = v$. Then since $v \in W$ and $a_{n+1} w_{n+1} \in W$, since W is a subspace, $v + a_{n+1} w_{n+1} \in W$. Ergo, (2) holds. This is enough to show (1).

The proof follows by induction.

7 Problem 1.3.23

Let W_1 and W_2 be subspaces of a vector space V.

1. Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.
2. Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

Proof: We'll start with (1). First, we want to show that $W_1 \subseteq W_1 + W_2$. Choose $x \in W_1$. Since W_2 is a subspace, $0 \in W_2$ where 0 is the zero vector of V. But $x = x + 0$ and $x \in W_1$. Thus, $x \in W_1 + W_2$ by definition. Ergo, $W_1 \subseteq W_1 + W_2$. We also must show that $W_2 \subseteq W_1 + W_2$, but this result is completely analogous (see if you can formalize it).

Now, we'll prove (2). Let X be a subspace of V such that $W_1, W_2 \subseteq X$. Choose $x \in W_1 + W_2$. Then $x = a + b$ for some $a \in W_1$ and $b \in W_2$. But by assumption then, $a, b \in X$. Since X is a subspace, $a + b \in X$. But $a + b = x$, so $x \in X$. This proves that $W_1 + W_2 \subseteq X$. Since X was arbitrary, we are done.

8 Problem 1.4.12

Show that a subset W of a vector space V is a subspace of V if and only if span $W = W$.

Proof: Suppose W is a subspace. Then choose a vector $v \in \text{span } W$. Then v can be written as,

$$v = \sum_{i=1}^{n} a_i w_i$$

for some n, $a_i \in F$ and $w_i \in W$. But W is a subspace, so it is closed under addition and scalar multiplication. Hence, $v \in W$. Now, choose $v \in W$. Then v is clearly in the span of W by definition of span. Hence, span $W = W$.

Now, suppose span $W = W$. The zero vector is in span W, so the zero vector is also in W. Suppose $a \in F$ and $v \in W$. Then since $av \in \text{span } W$, $av \in W$. Finally, suppose $v, w \in W$. Then $v + w \in \text{span } W$ so $v + w \in W$. We conclude that W is a subspace by definition.

Done.

9 Problem 1.5.2e

Show that the following set is linearly independent,

$$\{(1, -1, 2), (1, -2, 1), (1, 1, 4)\} \text{ in } \mathbb{R}^3$$

Solution: No, the set is not linearly independent. Simple calculations you learned in 33A will prove this. For example, set up the following matrix,

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & 1 \\ 2 & 1 & 4 \end{bmatrix}$$

Reducing this gives this matrix (if my arithmetic is correct):

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

which implies dependence. One could also take the determinant of this matrix and find that it is zero, which proves dependence. Of course, theoretically, you don't know why this is true yet!

10 Problem 1.5.17

Let M be a square upper triangular matrix with nonzero diagonal entries. Prove that the columns of M are linear independent.

Proof: Let M be $n \times n$. If $n = 1$, then the result is obvious (since the matrix must be nonzero by assumption). Suppose the n case holds and consider $n + 1$. The first n column vectors have an additional zero in their final entry, so they are still independent (check this!). Thus, adding the last column vector still results in an independent set (since the only way to kill the last entry is to multiply it by the zero vector). Hence, the $n + 1$ case holds and the induction is complete.

Note: this proof is fairly intuitive, so I've given a sketch. I'd like to see you fill in the blanks if you had trouble.