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Journal of
Statistical Theory and Practice
Volume 2, No.2, June 2008

Gambler's Ruin with Catastrophes and Windfalls

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Received : Sept. 15, 2006 *Revised* : Feb. 1, 2007

Abstract

We compute ruin probabilities, in both infinite-time and finite-time, for a Gambler's Ruin problem with both catastrophes and windfalls in addition to the customary win/loss probabilities. For constant transition probabilities, the infinite-time ruin probabilities are derived using difference equations. Finite-time ruin probabilities of a system having constant win/loss probabilities and variable catastrophe/windfall probabilities are determined using lattice path combinatorics. Formulae for expected time till ruin and the expected duration of gambling are also developed. The ruin probabilities (in infinite-time) for a system having variable win/loss/catastrophe probabilities but no windfall probability are found. Finally, the infinite-time ruin probabilities of a system with variable win/loss/catastrophe/windfall probabilities are determined.

AMS Subject Classification: 60J10

Key-words: Gambler's Ruin, Catastrophe, Birth-Death Process.

1. Introduction

The Gambler's Ruin problem is over 350 years old and has been traced back to letters between Blaise Pascal and Pierre Fermat; see, for example, Edwards (1983). In the traditional

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Gambler's Ruin problem, with two players P and Q having a total of H dollars between them, player P starts with j-dollars, $1 \leq j \leq H-1$, and makes a series of independent one dollar bets each having probability p of winning a dollar and probability q of losing a dollar. Player P's fortune at any point in time may be visualized as a state in the Markov chain diagram in Figure 1.

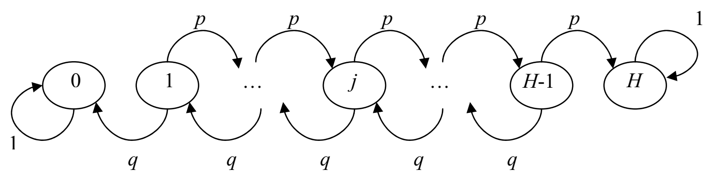

Figure 1. The state diagram of the traditional Gambler's Ruin problem

The game ends when player P loses all of his money or when he reaches his goal of winning H dollars. The objective is to determine P's ruin probability, that is, the chance of reaching state 0 assuming player P begins with j-dollars. The most well-known version of this problem allows play to continue indefinitely until player P reaches state 0 or H. A related problem asks for the ruin probabilities allowing only, at most, a specified, finite number of bets. Solutions to both versions of the gambler's ruin problem date back at least to the late 1600's. In addition to Pascal and Fermat, solutions were obtained by C. Huygens, J. Bernoulli, A. de Moivre, P. de Montmort and N. Bernoulli; see, for example, Edwards (1983) and Takacs (1969). Nicely presented discussions of Gambler's Ruin problems may also be found in the books by Ash (1970), Feller (1968), and Hoel, Port, and Stone (1987). An interesting article by Harper and Ross (2005) reports some recent extensions of gambler's ruin probabilities to different state transition diagrams.

In this article, some natural generalizations of classical Gambler's Ruin problems are considered. In Sections 2 and 3, the problem of determining ruin probabilities is solved when constant catastrophe/windfall probabilities occur in addition to the customary, constant win/loss probabilities. In Section 2, the infinite-time, ruin probability problem for the chain of Figure 2 is solved using difference equations while in Section 3, lattice path combinatorics is utilized to obtain ruin probabilities whenever a specific, finite number of transitions is allowed in Figure 2. Comparisons between finite-time and infinite-time ruin probabilities are graphed. By taking the limit of these finite-time ruin probabilities, a trigonometric expression for the infinite-time ruin probabilities is obtained. It is interesting to contrast these apparently dissimilar (yet equivalent) formulae for the infinite-time ruin probabilities obtained by the different methods described in Sections 2 and 3.

In Section 4, ruin probabilities (in infinite-time) of a system having variable win/loss/catastrophe probabilities but no windfall probabilities are found. In Section 5, the distribution functions of the time at which ruin occurs, when state H is reached or when play terminates, presuming that gambling began from state j, are presented. Formulae for the expected time of ruin conditioned upon ruin occurring and the expected duration of play are derived and tabulated. In Section 6, the infinite-time ruin probabilities of a system having variable win/loss/catastrophe/windfall probabilities are determined. The recursive

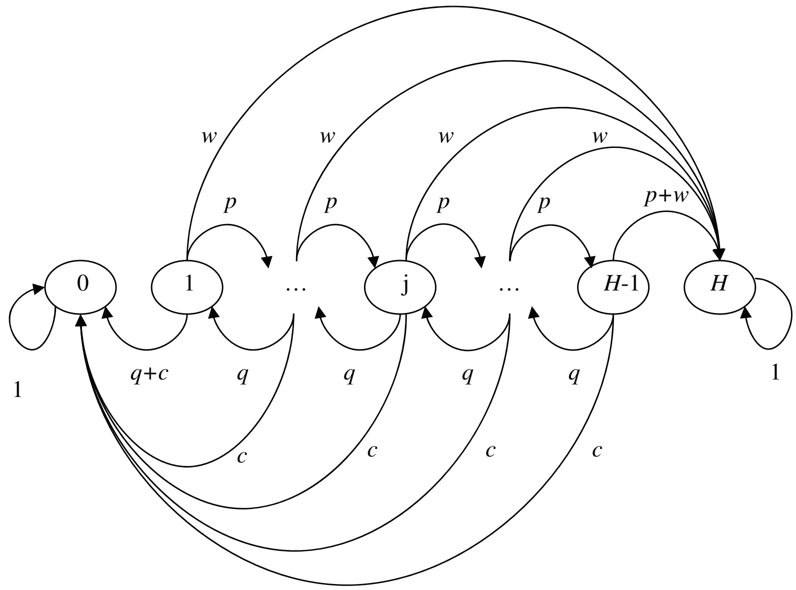

Figure 2. Transition state diagram of Gambler's Ruin with constant catastrophe and windfall probabilities

approaches in Sections 4 and 6 generalize the known ruin probability formula in Hoel, Port, and Stone (1987) for general birth-death chains. If the systems described in Sections 2, 4 and 6 are changed to include return transition probabilities (loops), then the same recursive methods, suitably modified, still produce formulae for computing ruin probabilities. This result and some related problems and directions for further study are mentioned in Section 7.

2. Infinite-time Gambler's ruin probabilities with catastrophes and windfalls

Suppose that in each round of a Gambler's Ruin game player P wins with probability p, loses with probability q, experiences a catastrophe taking him or her to state 0 with probability c, and gains a windfall taking him or her to state H with probability w, where $p+q+c+w=1$, as shown in the Markov chain state diagram in Figure 2.

Let infinite-time ruin probabilities be given by

$$r_k = \text{Prob}(P \text{ is eventually ruined} \mid P \text{ is initially at state } k).$$

Suppose that player P currently has k dollars, where $1 \leq k \leq H-1$. Conditioning on the next round of play, with probability c player P will go directly to state 0 and be ruined; with probability q player P will go to state $k-1$, from which he or she has probability r_{k-1} of eventual ruin; with probability p player P will go to state $k+1$, from which he or she has probability r_{k+1} of eventual ruin; and with probability w player P will go directly to state H, from which he or she has no chance of being ruined.

Thus, for $1 \le k \le H-1$,

$$\begin{aligned} r_k &= cr_0 + qr_{k-1} + pr_{k+1} + wr_H \\ &= c + qr_{k-1} + pr_{k+1}, \end{aligned}$$

where we have used the fact that $r_0 = 1$ (since the player is already ruined in this case) and $r_H = 0$ (since upon reaching state H the player stops playing the game). That is,

$$\begin{aligned} r_0 &= 1 \\ r_1 &= c + qr_0 + pr_2 \\ r_2 &= c + qr_1 + pr_3 \\ &\vdots \\ r_k &= c + qr_{k-1} + pr_{k+1} \\ &\vdots \\ r_{H-1} &= c + qr_{H-2} + pr_H \\ r_H &= 0. \end{aligned} \tag{2.1}$$

This is a set of linear constant-coefficient difference equations with "boundary values" $r_0 = 1$ and $r_H = 0$. In other words, we are looking for the solution of

$$pr_{k+1} - r_k + qr_{k-1} = -c \tag{2.2}$$

subject to the "boundary values" $r_0 = 1$ and $r_H = 0$. The general solution of (2.2) may be found as a sum of the general solution of the associated homogeneous equation

$$pr_{k+1} - r_k + qr_{k-1} = 0 \tag{2.3}$$

and a particular solution of the non-homogeneous equation (2.2), see Goldberg (1986) and Marcus (1998).

To find a particular solution of (2.2), we will assume $r_k = A$ where A is a constant. Then substituting into (2.2),

$$pA - A + qA = -c$$

and we obtain $r_k = \frac{c}{c+w}$, for all k.

For the homogeneous equation (2.3), the characteristic polynomial is $px^2 - x + q$. The roots of the characteristic polynomial are $(1 \pm \sqrt{1-4pq})/(2p)$. Thus, for $pq \neq 1/4$, the general solution of the non-homogeneous equation (2.2) is

$$r_k = C_1 \left[\frac{1+\sqrt{1-4pq}}{2p}\right]^k + C_2 \left[\frac{1-\sqrt{1-4pq}}{2p}\right]^k + \frac{c}{c+w}. \tag{2.4}$$

Applying the initial conditions, $r_0 = 1$ and $r_H = 0$, we have that the ruin probabilities (for $pq \neq \frac{1}{4}$) are given by

$$r_k = \frac{c}{c+w} + \left[-\frac{c(2p)^H + w(1-\sqrt{1-4pq})^H}{(c+w)[(1+\sqrt{1-4pq})^H - (1-\sqrt{1-4pq})^H]}\right] \cdot \left[\frac{1+\sqrt{1-4pq}}{2p}\right]^k$$

$$+\left[1-\frac{c}{c+w}+\frac{c(2p)^H+w(1-\sqrt{1-4pq})^H}{(c+w)[(1+\sqrt{1-4pq})^H-(1-\sqrt{1-4pq})^H]}\right]\cdot\left[\frac{1-\sqrt{1-4pq}}{2p}\right]^k. \quad (2.5)$$

In the special case of catastrophes but no windfalls ($w=0$), again assuming $pq \neq \frac{1}{4}$, this result reduces to

$$r_k = 1-\frac{\left[\frac{1+\sqrt{1-4pq}}{2p}\right]^k-\left[\frac{1-\sqrt{1-4pq}}{2p}\right]^k}{\left[\frac{1+\sqrt{1-4pq}}{2p}\right]^H-\left[\frac{1-\sqrt{1-4pq}}{2p}\right]^H}, \quad \text{for } w=0. \qquad (2.6)$$

We can use the Binomial Theorem to re-write this $w=0$ result without using radicals:

$$\begin{aligned}
\left(1+\sqrt{x}\right)^k &= \sum_{i=0}^{k}\binom{k}{i}\left(\sqrt{x}\right)^i \\
\left(1-\sqrt{x}\right)^k &= \sum_{i=0}^{k}(-1)^i\binom{k}{i}\left(\sqrt{x}\right)^i \\
\left(1+\sqrt{x}\right)^k-\left(1-\sqrt{x}\right)^k &= 2\sum_{\substack{i=1\\ i\text{ odd}}}^{k}\binom{k}{i}\left(\sqrt{x}\right)^i \\
&= 2\sqrt{x}\sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor}\binom{k}{2j+1}x^j
\end{aligned}$$

where $\lfloor y \rfloor$ denotes the greatest integer $\leq y$. The resulting alternative form for the ruin probabilities with $w=0$ (and still assuming $pq \neq 1/4$) is

$$r_k = 1-\left[(2p)^{H-k}\cdot\frac{\sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor}\binom{k}{2j+1}(1-4pq)^j}{\sum_{j=0}^{\lfloor\frac{H-1}{2}\rfloor}\binom{H}{2j+1}(1-4pq)^j}\right], \quad \text{for } w=0,\ pq\neq 1/4. \qquad (2.7)$$

In the special case of no catastrophes or windfalls ($c=w=0$), for the moment still assuming $pq \neq \frac{1}{4}$, the result in (2.6) reduces even further. The roots of the characteristic polynomial become $\frac{q}{p}$ and 1, and we obtain ruin probabilities

$$r_k = \frac{\left(\frac{q}{p}\right)^H-\left(\frac{q}{p}\right)^k}{\left(\frac{q}{p}\right)^H-1}, \qquad \text{for } c=w=0,\ pq\neq\frac{1}{4}. \qquad (2.8)$$

Finally, in the special case in which $p=q=\frac{1}{2}$, the characteristic polynomial has a double root of 1. In this case, the general solution has the form

$$r_k = C_1 + C_2 k. \qquad (2.9)$$

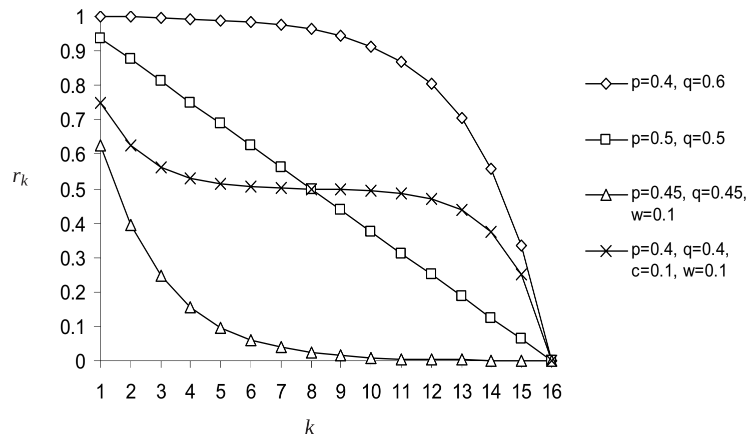

Figure 3. Infinite-time ruin probabilities with $H = 16$ for several sets of p, q, c, w values with $p+q+c+w = 1$

Applying the initial conditions, we obtain ruin probabilities

$$r_k = 1 - \frac{k}{H}, \qquad \text{for } \ p = q = \frac{1}{2}. \tag{2.10}$$

We have thus fully characterized the ruin probabilities in infinite time with constant-rate catastrophes and windfalls.

In Figure 3, we illustrate infinite-time ruin probabilities with $H = 16$ for several sets of p, q, c, w values with $p+q+c+w = 1$. Note the effects of loss of symmetry as shown in the uppermost and lowest curves, as well as the effects of introducing catastrophes and windfalls in a symmetric manner as shown in the middle non-linear curve. We turn next to ruin probabilities in finite time.

3. Finite-time ruin probabilities with catastrophes and windfalls

Let finite-step ruin probabilities be given by

$$r_k^{(n)} = \text{Prob}(P \text{ is ruined in first } n \text{ steps} \mid P \text{ is initially at state } k).$$

In this section, we continue to assume that $p+q+c+w = 1$.

Let

$$L_{k,j}^{(n)} = \text{number of paths from state } k \text{ to state } j \text{ in } n \text{ steps},$$

where $1 \leq j \leq H-1$ and $1 \leq k \leq H-1$ and the paths are restricted from hitting the absorbing states at 0 and H. It is shown in Mohanty (1979) and Narayana (1979) that

$$L_{k,j}^{(n)} = \sum_{i=1}^{H+1} \left[\binom{n}{\frac{n-j+k}{2} - i(H+1)} - \binom{n}{\frac{n+j+k-2}{2} + i(H+1) + 1} \right] \tag{3.1a}$$

and in Krinik *et al.* (2005) and Takacs (1969) that

$$L_{k,j}^{(n)} = \frac{2}{H}\sum_{u=1}^{H}\sin\left(\frac{u\pi j}{H}\right)\sin\left(\frac{u\pi k}{H}\right)\left[2\cos\left(\frac{u\pi}{H}\right)\right]^n. \tag{3.1b}$$

We use the convention that binomial coefficients $\binom{a}{b}$ are defined to be 0 if $b > a$ or $b < 0$ or if neither a nor b is integer-valued.

While it is not even obvious that (3.1b) is integer-valued, it does in fact count the number of paths.

Let

$$P_{k,j}^{(n)} = \text{Prob}(P \text{ is at state } j \text{ at } n\text{th step} \mid P \text{ is initially at state } k).$$

Assuming $1 \le j \le H-1$ and $1 \le k \le H-1$,

$$P_{k,j}^{(n)} = L_{k,j}^{(n)} p^{(n+j-k)/2} q^{(n-j+k)/2}, \tag{3.2}$$

where the exponents on p and q ensure that n steps are taken in such a way that the net change in position is $j-k$. That is, $(n+j-k)/2 + (n-j+k)/2 = n$ and $(n+j-k)/2 - (n-j+k)/2 = j-k$. Substituting (3.1b) into (3.2),

$$P_{k,j}^{(n)} = \frac{2}{H} p^{(j-k)/2} q^{(k-j)/2} \sum_{u=1}^{H}\sin\left(\frac{u\pi j}{H}\right)\sin\left(\frac{u\pi k}{H}\right)\left[2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right]^n \tag{3.3}$$

for $1 \le j \le H-1$ and $1 \le k \le H-1$.

Now, for $n > 0$,

$$r_k^{(n)} = \sum_{i=0}^{n-1} P_{k,1}^{(i)} \cdot q + \sum_{i=0}^{n-1}\sum_{j=1}^{H-1} P_{k,j}^{(i)} \cdot c. \tag{3.4}$$

This follows from the fact that in order to reach state 0, either a) after some number of steps we reached state 1 and then took an immediate step down to state 0, or b) after some number of steps we reached state j and then experienced a catastrophe taking us to state 0; see Hunter (2005) for more details. Using (3.3) in (3.4) we obtain, for $1 \le k \le H-1$, the finite-time ruin probabilities

$$\begin{aligned} r_k^{(n)} = {} & \sum_{i=0}^{n-1} \frac{2}{H} p^{(i-k+1)/2} q^{(i+k-1)/2} \sum_{u=1}^{H}\sin\left(\frac{u\pi}{H}\right)\sin\left(\frac{u\pi k}{H}\right)\left[2\cos\left(\frac{u\pi}{H}\right)\right]^i \cdot q \\ & + \sum_{i=0}^{n-1}\sum_{j=1}^{H-1} \frac{2}{H} p^{(i+j-k)/2} q^{(i+k-j)/2} \sum_{u=1}^{H}\sin\left(\frac{u\pi j}{H}\right)\sin\left(\frac{u\pi k}{H}\right)\left[2\cos\left(\frac{u\pi}{H}\right)\right]^i \cdot c \end{aligned} \tag{3.5}$$

For completeness we may also give the trivial results $r_0^{(n)} = 1$ and $r_H^{(n)} = 0$.

In the limit as $n \to \infty$, the result in (3.5) should reduce to the result in (2.5). While this appears difficult to prove analytically, in what follows we derive one form for (3.5) in the limit as $n \to \infty$. We assume $pq \ne \frac{1}{4}$ (that is, $pq < \frac{1}{4}$ since $p+q+c+w = 1$) as this was assumed in (2.5).

As a first step, we change the order of summations and rearrange factors in (3.5):

$$r_k^{(n)} = \frac{2\sqrt{pq}}{H}\left(\sqrt{\frac{q}{p}}\right)^k \cdot \sum_{u=1}^{H} \sin\left(\frac{u\pi}{H}\right)\sin\left(\frac{u\pi k}{H}\right)\sum_{i=0}^{n-1}\left[2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right]^i$$
$$+\frac{2c}{H}\left(\sqrt{\frac{q}{p}}\right)^k \cdot \sum_{j=1}^{H-1}\left(\sqrt{\frac{p}{q}}\right)^j \sum_{u=1}^{H} \sin\left(\frac{u\pi j}{H}\right)\sin\left(\frac{u\pi k}{H}\right)\sum_{i=0}^{n-1}\left[2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right]^i \quad (3.6)$$

Now, in the limit as $n \to \infty$, the summation with respect to i is a convergent geometric series since $2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right) < 1$ for $pq \neq \frac{1}{4}$. Thus, we obtain a new form for the infinite-time ruin probabilities for $1 \leq k \leq H-1$ with $pq \neq \frac{1}{4}$:

$$\begin{aligned} r_k &= \lim_{n\to\infty} r_k^{(n)} \\ &= \frac{2\sqrt{pq}}{H}\left(\sqrt{\frac{q}{p}}\right)^k \cdot \sum_{u=1}^{H} \frac{\sin\left(\frac{u\pi}{H}\right)\sin\left(\frac{u\pi k}{H}\right)}{1-2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)} \\ &\quad +\frac{2c}{H}\left(\sqrt{\frac{q}{p}}\right)^k \cdot \sum_{j=1}^{H-1}\left(\sqrt{\frac{p}{q}}\right)^j \sum_{u=1}^{H} \frac{\sin\left(\frac{u\pi j}{H}\right)\sin\left(\frac{u\pi k}{H}\right)}{1-2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)} \end{aligned} \quad (3.7)$$

We have confirmed numerically that (3.7) is equivalent to (2.5).

In the example illustrated in Figure 4, we see that the finite-time ruin probabilities with catastrophes and windfalls (equation (3.5)) converge to the infinite-time ruin probabilities (equation (2.5)) as the number of steps, n, is increased. In the example we let $p = 0.1$, $q = 0.65$, $c = 0.13$, $w = 0.12$ and $H = 16$. Figure 4 shows the computed finite and infinite-time ruin probabilities for the above-mentioned parameters. The finite ruin probabilities are shown for number of steps of $2, 4, 6, 8, 10$ and 20. As the number of steps increases, the finite-time ruin probabilities converge to the infinite-time ruin probabilities. Within the resolution of the graph, $r_k^{(20)}$ and r_k are indistinguishable.

4. Infinite-time Gambler's ruin probabilities with variable win/loss/catastrophe probabilities and no windfalls

Suppose now that the transition probabilities are state-dependent with p_k, q_k, c_k, and w_k representing probabilities associated with state k. Suppose also that there are no windfalls ($w_k = 0$ for all k) so that for $1 \leq k \leq H-1$, $p_k + q_k + c_k = 1$. The corresponding Markov chain diagram is shown in Figure 5.

As before, let infinite-time ruin probabilities be given by

$$r_k = \text{Prob}(P \text{ is eventually ruined} \mid P \text{ is initially at state } k).$$

Also define "success" probabilities by

$$s_k = \text{Prob}(P \text{ is eventually absorbed at state } H \mid P \text{ is initially at state } k) = 1 - r_k.$$

The difference equations corresponding to r_k or s_k no longer involve constant transition

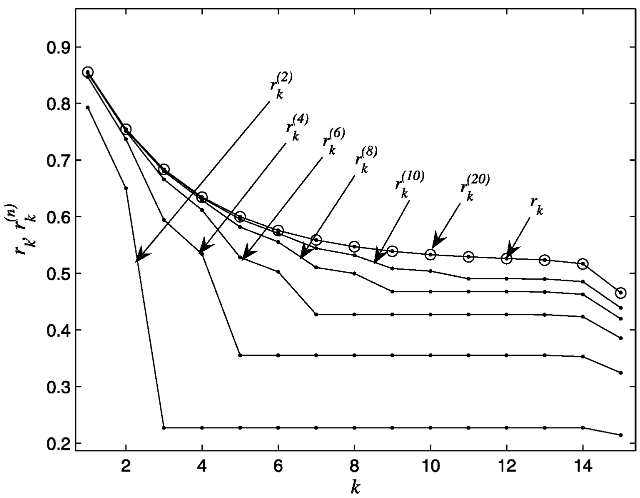

Figure 4. Finite and Infinite-time ruin probabilities

probabilities. Nevertheless, we will be able to find an expression for s_k, and $r_k = 1 - s_k$.

In what follows we will prove that, for $1 \leq k \leq H$,

$$\begin{aligned} s_k \;=\; & \left[1 - \sum_{i=1}^{k-2} p_i q_{i+1} + \sum_{i=1}^{k-4} p_i q_{i+1} \left[\sum_{j=i}^{k-4} p_{j+2} q_{j+3} \right] \right. \\ & \left. - \sum_{i=1}^{k-6} p_i q_{i+1} \left[\sum_{j=i}^{k-6} p_{j+2} q_{j+3} \left[\sum_{l=j}^{k-6} p_{l+4} q_{l+5} \right] \right] + \ldots \right] s_1 \Big/ \left[\prod_{i=1}^{k-1} p_i \right], \end{aligned} \tag{4.1}$$

with the convention that summations are set equal to zero if they have an upper limit of summation that is less than the lower limit and products are set equal to one if they have an upper limit that is less than the lower limit.

Multiplying equation (4.1) by $\prod_{i=1}^{k-1} p_i$,

$$\begin{aligned} s_k \cdot \prod_{i=1}^{k-1} p_i \;=\; & s_1 \cdot \left[1 - \sum_{i=1}^{k-2} p_i q_{i+1} + \sum_{i=1}^{k-4} p_i q_{i+1} \left[\sum_{j=i}^{k-4} p_{j+2} q_{j+3} \right] \right. \\ & \left. - \sum_{i=1}^{k-6} p_i q_{i+1} \left[\sum_{j=i}^{k-6} p_{j+2} q_{j+3} \left[\sum_{l=j}^{k-6} p_{l+4} q_{l+5} \right] \right] + \ldots \right]. \end{aligned}$$

Solving for s_1,

$$\begin{aligned} s_1 \;=\; & s_k \cdot \prod_{i=1}^{k-1} p_i + s_1 \cdot \left[\sum_{i=1}^{k-2} p_i q_{i+1} - \sum_{i=1}^{k-4} p_i q_{i+1} \left[\sum_{j=i}^{k-4} p_{j+2} q_{j+3} \right] \right. \\ & \left. + \sum_{i=1}^{k-6} p_i q_{i+1} \left[\sum_{j=i}^{k-6} p_{j+2} q_{j+3} \left[\sum_{l=j}^{k-6} p_{l+4} q_{l+5} \right] \right] - \ldots \right]. \end{aligned} \tag{4.2}$$

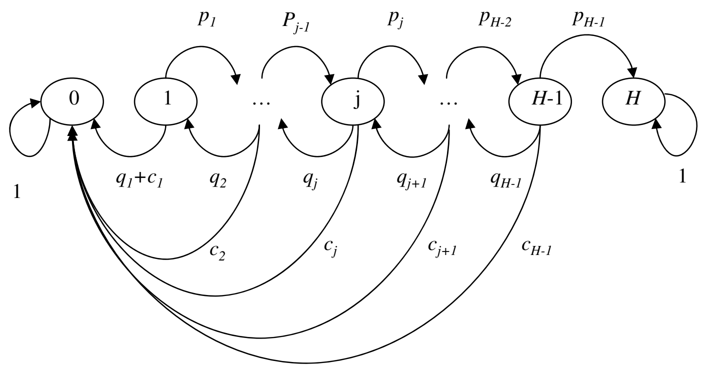

Figure 5. Transition state diagram of Gambler's Ruin with variable win, loss, and catastrophe probabilities and no windfalls

We will prove (4.1) by first proving (4.2). The quantity s_1 on the left-hand side of (4.2) is the probability of eventual success starting from state 1, where "success" consists of reaching state H. We will prove that the right hand side of (4.2) also gives the probability of eventual success starting from state 1.

Let k be fixed, with $1 \leq k \leq H$. The term

$$s_k \cdot \prod_{i=1}^{k-1} p_i$$

gives the probability of eventual success from state 1 with only upward motion before reaching state k. To see this, note that the product $\prod_{i=1}^{k-1} p_i$ is the probability of "walking" directly from state 1 to state 2, …, to state k by winning each of the first $k-1$ rounds. The factor s_k then gives the probability of eventual success starting from state k.

The other terms on the right hand side of (4.2) give the probability of eventual success from state 1 with one or more "up-and-down" motions before reaching state k. We are defining "up-and-down from state i," with $1 \leq i \leq k-2$, as movement from state i directly to state $i+1$ and directly back to state i. That is, "up and down" is a win immediately followed by a loss.

For $1 \leq i \leq k-2$, the probability that a randomly-chosen path is eventually successful but involves movement from the first occurrence of state i directly to state $i+1$ and directly back to state i is given by $s_1 p_i q_{i+1}$, a type of product that appears in many places in (4.2). To see this, we will look at several equivalent probabilities. Note that the following three probabilities are equivalent:

(a) Prob(starting from state 1, player P reaches state i for the first time by some means, then does an "up-and-down," and continues, eventually reaching state H)

(b) Prob(starting from state 1, player P reaches state i for the first time by some means and from there reaches state H by some means)$\cdot$ Prob(from the first occurrence of state i, the next two steps are "up-and-down")

(c) Prob(starting from state 1, player P reaches H by some means)$\cdot$ Prob(when player P reaches state i for the first time, the next two steps are "up-and-down")

Note that (a), (b), and (c) are all equal to $s_1 p_i q_{i+1}$ since s_1 is the probability of eventual successful movement to state H from initial state 1 and $p_i q_{i+1}$ is the probability that from the first occurrence of state i the next two steps are "up-and-down."

Now the sum multiplying s_1 on the right-hand side of (4.2),

$$\sum_{i=1}^{k-2} p_i q_{i+1} - \sum_{i=1}^{k-4} p_i q_{i+1} \left[\sum_{j=i}^{k-4} p_{j+2} q_{j+3} \right] + \sum_{i=1}^{k-6} p_i q_{i+1} \left[\sum_{j=i}^{k-6} p_{j+2} q_{j+3} \left[\sum_{l=j}^{k-6} p_{l+4} q_{l+5} \right] \right] - \dots,$$

is simply an application of the Inclusion/Exclusion Principle giving the probability of eventual success from state 1 with some "up-and-down" motion before reaching state k, where k is fixed with $1 \le k \le H$. The first summation in this sum,

$$\sum_{i=1}^{k-2} p_i q_{i+1},$$

gives the sum of the probabilities that from state i, the next two steps are "up-and-down," for states i from 1 to $k-2$. (If we let i reach $k-1$, the "up-and-down" would take us to state k on its "up," resulting in this "up-and-down" not being completed before the first occurrence of state k.) However, for non-adjacent states i, j, it is possible for a path to success to have an "up-and-down" motion at the first occurrence of both state i and state j. The double summation

$$\sum_{i=1}^{k-4} p_i q_{i+1} \left[\sum_{j=i}^{k-4} p_{j+2} q_{j+3} \right]$$

is subtracted, since it is the sum of probabilities of "up-and-down" motions at the first occurrences of two different (non-adjacent) states. The upper limit of summation has been decreased to $k-4$ to allow "space" for both "up-and-downs" to occur at states $\le k-2$. The triple summation

$$\sum_{i=1}^{k-6} p_i q_{i+1} \left[\sum_{j=i}^{k-6} p_{j+2} q_{j+3} \left[\sum_{l=j}^{k-6} p_{l+4} q_{l+5} \right] \right]$$

then is added back, since it is the sum of probabilities of "up-and-downs" occurring at the first occurrences of three different (pairwise non-adjacent) states. This Inclusion/Exclusion is continued in order to compute correctly a quantity that, when multiplied by s_1, gives the probability of eventual success from state 1 with some "up-and-down" motions before reaching state k for the first time. Adding the first term on the right hand side of (4.2), we obtain the total probability of success from initial state 1, with or without "up-and-down" motions before reaching state k for the first time, as given on the left-hand side of (4.2). This completes the proof of (4.2), and thus also of (4.1).

Now, setting $k = H$ in (4.1) and recalling that $s_H = 1$,

$$\begin{aligned} s_H &= \left[1 - \sum_{i=1}^{H-2} p_i q_{i+1} + \sum_{i=1}^{H-4} p_i q_{i+1} \left[\sum_{j=i}^{H-4} p_{j+2} q_{j+3}\right]\right. \\ &\quad \left. - \sum_{i=1}^{H-6} p_i q_{i+1} \left[\sum_{j=i}^{H-6} p_{j+2} q_{j+3} \left[\sum_{l=j}^{H-6} p_{l+4} q_{l+5}\right]\right] + \ldots\right] s_1 \Big/ \left[\prod_{i=1}^{H-1} p_i\right] \\ &= 1. \end{aligned}$$

This allows us to solve for s_1, obtaining

$$\begin{aligned} s_1 &= \left[\prod_{i=1}^{H-1} p_i\right] \Big/ \left[1 - \sum_{i=1}^{H-2} p_i q_{i+1} + \sum_{i=1}^{H-4} p_i q_{i+1} \left[\sum_{j=i}^{H-4} p_{j+2} q_{j+3}\right]\right. \\ &\quad \left. - \sum_{i=1}^{H-6} p_i q_{i+1} \left[\sum_{j=i}^{H-6} p_{j+2} q_{j+3} \left[\sum_{l=j}^{H-6} p_{l+4} q_{l+5}\right]\right] + \ldots\right]. \end{aligned}$$

Finally, this in turn implies by (4.1) that

$$s_k = \left[\frac{1 - \sum_{i=1}^{k-2} p_i q_{i+1} + \sum_{i=1}^{k-4} p_i q_{i+1} \left[\sum_{j=i}^{k-4} p_{j+2} q_{j+3}\right] - \ldots}{1 - \sum_{i=1}^{H-2} p_i q_{i+1} + \sum_{i=1}^{H-4} p_i q_{i+1} \left[\sum_{j=i}^{H-4} p_{j+2} q_{j+3}\right] - \ldots}\right] \cdot \prod_{i=k}^{H-1} p_i.$$

The ruin probabilities, for $1 \leq k \leq H$, are therefore given by

$$r_k = 1 - \left[\frac{1 - \sum_{i=1}^{k-2} p_i q_{i+1} + \sum_{i=1}^{k-4} p_i q_{i+1} \left[\sum_{j=i}^{k-4} p_{j+2} q_{j+3}\right] - \ldots}{1 - \sum_{i=1}^{H-2} p_i q_{i+1} + \sum_{i=1}^{H-4} p_i q_{i+1} \left[\sum_{j=i}^{H-4} p_{j+2} q_{j+3}\right] - \ldots}\right] \cdot \prod_{i=k}^{H-1} p_i.$$

We turn next to questions of duration of play.

5. Duration of play in Gambler's ruin with variable catastrophe and windfall probabilities

Let R_k be a random variable denoting the trial on which a player is ruined assuming the player starts at state k. Then

$$P\left[R_k = n\right] = q P_{k,1}^{(n-1)} + \sum_{j=1}^{H-1} c P_{k,j}^{(n-1)} \qquad \text{for } n = 1, 2, 3, \ldots.$$

If we assume now that the catastrophe and windfall probabilities are variable, with $p + q + c_i + w_i = 1$, for $i = 1, 2, \ldots, H-1$, as shown in Figure 6, then

$$P\left[R_k = n\right] = q P_{k,1}^{(n-1)} + \sum_{j=1}^{H-1} c_j P_{k,j}^{(n-1)} \qquad \text{for } n = 1, 2, 3, \ldots. \tag{5.1}$$

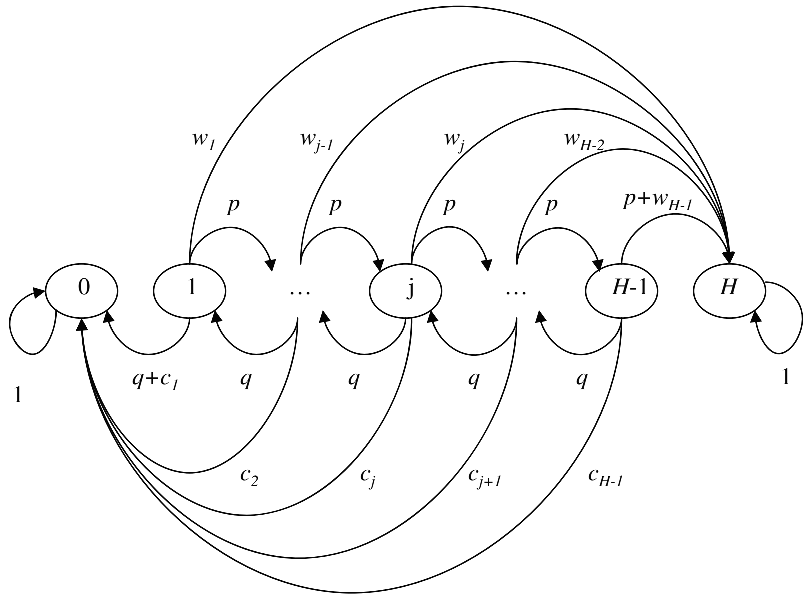

Figure 6. Transition state diagram of Gambler's Ruin with constant win and loss probabilities and variable catastrophe and windfall probabilities

and if S_k is the random variable representing the time when one reaches H assuming k is the starting point, then

$$P\left[S_k = n\right] = pP_{k,H-1}^{(n-1)} + \sum_{j=1}^{H-1} w_j P_{k,j}^{(n-1)}. \tag{5.2}$$

Suppose D_k represents the duration time of the gambling. That is, D_k is the trial when the gambling terminates assuming we began at state k. Then

$$P\left[D_k = n\right] = P\left[R_k = n\right] + P\left[S_k = n\right]. \tag{5.3}$$

Now

$$r_k = \sum_{n=1}^{\infty} P\left[R_k = n\right] = q\sum_{n=1}^{\infty} P_{k,1}^{(n-1)} + \sum_{j=1}^{H-1} c_j \sum_{n=1}^{\infty} P_{k,j}^{(n-1)}.$$

Proceeding as in the derivation of (3.7), this time with variable catastrophe probabilities,

$$\begin{aligned} r_k \;=\; & \frac{2\sqrt{pq}}{H}\left(\sqrt{\frac{q}{p}}\right)^k \cdot \sum_{u=1}^{H} \frac{\sin\left(\frac{u\pi}{H}\right)\sin\left(\frac{u\pi k}{H}\right)}{1-2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)} \\ & + \frac{2}{H}\left(\sqrt{\frac{q}{p}}\right)^k \cdot \sum_{j=1}^{H-1} c_j \left(\sqrt{\frac{p}{q}}\right)^j \sum_{u=1}^{H} \frac{\sin\left(\frac{u\pi j}{H}\right)\sin\left(\frac{u\pi k}{H}\right)}{1-2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)}. \end{aligned} \tag{5.4}$$

Note that (5.4) still applies if $p = q = 1/2$.

Similarly,

$$\begin{aligned} s_k &= \sum_{n=1}^{\infty} P\left[S_k = n\right] \\ &= p\sum_{n=1}^{\infty} P_{k,H-1}^{(n-1)} + \sum_{j=1}^{H-1} w_j \sum_{n=1}^{\infty} P_{k,j}^{(n-1)}. \end{aligned}$$

After a derivation similar to that used in obtaining (3.7), we find

$$\begin{aligned} s_k &= \frac{2\sqrt{pq}}{H}\left(\sqrt{\frac{q}{p}}\right)^k \cdot \left(\sqrt{\frac{p}{q}}\right)^H \cdot \sum_{u=1}^{H} \frac{\sin\left(\frac{u\pi(H-1)}{H}\right)\sin\left(\frac{u\pi k}{H}\right)}{1-2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)} \\ &\quad + \frac{2}{H}\left(\sqrt{\frac{q}{p}}\right)^k \cdot \sum_{j=1}^{H-1} w_j \left(\sqrt{\frac{p}{q}}\right)^j \sum_{u=1}^{H} \frac{\sin\left(\frac{u\pi j}{H}\right)\sin\left(\frac{u\pi k}{H}\right)}{1-2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)}. \end{aligned} \tag{5.5}$$

Note that if $p+q+c_i+w_i = 1$ for $i = 1, 2, \ldots, H-1$, then $1-r_k = s_k$. However, equations (5.1), (5.2), (5.3), (5.4), (5.5) still hold in Figure 6 when $p+q+c_i+w_i < 1$ for $i = 1, 2, \ldots, H-1$. We note again that when $c_i = c$ and $w_i = w$ for $i = 1, 2, \ldots, H-1$, and $p+q+c+w = 1$, equation (5.4) provides an alternative expression for r_k found (in Section 2) by the difference equation approach leading to equation (2.5).

An interesting question concerns the expected time of ruin assuming a gambler is going to be ruined eventually. This conditional expectation may be computed as follows:

$$\begin{aligned} E\left[R_k|\text{eventual ruin}\right] &= \sum_{n=1}^{\infty} nP\left[R_k = n|\text{eventual ruin}\right] \\ &= \sum_{n=1}^{\infty} n\frac{P\left[R_k = n\right]}{r_k} \\ &= \frac{1}{r_k}\sum_{n=1}^{\infty} n\left[qP_{k,1}^{(n-1)} + \sum_{j=1}^{H-1} c_j P_{k,j}^{(n-1)}\right] \end{aligned}$$

where we have used equation (5.1). Continuing in a manner similar to that used in the derivation of equations (3.7) and (5.4),

$$\begin{aligned} E\left[R_k|\text{eventual ruin}\right] &= \frac{2\sqrt{pq}}{H}\left(\sqrt{\frac{q}{p}}\right)^k \cdot \frac{1}{r_k} \cdot \sum_{u=1}^{H} \sin\left(\frac{u\pi}{H}\right)\sin\left(\frac{u\pi k}{H}\right) \\ &\quad \cdot \sum_{n=1}^{\infty} n\left[2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right]^{n-1} \\ &\quad + \frac{2}{H}\left(\sqrt{\frac{q}{p}}\right)^k \cdot \frac{1}{r_k} \cdot \sum_{j=1}^{H-1} c_j \left(\sqrt{\frac{p}{q}}\right)^j \sum_{u=1}^{H} \sin\left(\frac{u\pi j}{H}\right)\sin\left(\frac{u\pi k}{H}\right) \\ &\quad \cdot \sum_{n=1}^{\infty} n\left[2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right]^{n-1}. \end{aligned}$$

But $\frac{1}{1-x} = \sum_{n=1}^{\infty} x^n$ implies $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$ for $|x| < 1$. Thus,

$$\begin{aligned}
&E\left[R_k|\text{eventual ruin}\right] \\
&\quad = \frac{2\sqrt{pq}}{H}\left(\sqrt{\frac{q}{p}}\right)^k \cdot \frac{1}{r_k} \cdot \sum_{u=1}^{H} \frac{\sin\left(\frac{u\pi}{H}\right)\sin\left(\frac{u\pi k}{H}\right)}{\left(1-2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right)^2} \\
&\qquad + \frac{2}{H}\left(\sqrt{\frac{q}{p}}\right)^k \cdot \frac{1}{r_k} \cdot \sum_{j=1}^{H-1} c_j \left(\sqrt{\frac{p}{q}}\right)^j \sum_{u=1}^{H} \frac{\sin\left(\frac{u\pi j}{H}\right)\sin\left(\frac{u\pi k}{H}\right)}{\left(1-2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right)^2}. \qquad (5.6)
\end{aligned}$$

A similar formula may be derived for $E[S_k|\text{eventual success}]$ and conditional variances may also be computed. The average duration $E[D_k]$ may now be calculated as

$$\begin{aligned}
E[D_k] &= \sum_{n=1}^{\infty} nP[D_k = n] \\
&= \sum_{n=1}^{\infty} n\left(P\left[R_k = n\right] + P\left[S_k = n\right]\right) \\
&= \sum_{n=1}^{\infty} n\left(qP_{k,1}^{(n-1)} + \sum_{j=1}^{H-1} c_j P_{k,j}^{(n-1)}\right) \\
&\quad + \sum_{n=1}^{\infty} n\left(pP_{k,H-1}^{(n-1)} + \sum_{j=1}^{H-1} w_j P_{k,j}^{(n-1)}\right)
\end{aligned}$$

where we have used equation (5.3). After a derivation similar to that used to obtain (5.6), we find that

$$\begin{aligned}
E\left[D_k\right] &= \frac{2\sqrt{pq}}{H}\left(\sqrt{\frac{q}{p}}\right)^k \cdot \sum_{u=1}^{H} \frac{\sin\left(\frac{u\pi}{H}\right)\sin\left(\frac{u\pi k}{H}\right)}{\left(1-2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right)^2} \\
&\quad + \frac{2\sqrt{pq}}{H}\left(\sqrt{\frac{q}{p}}\right)^k \cdot \left(\sqrt{\frac{p}{q}}\right)^H \cdot \sum_{u=1}^{H} \frac{\sin\left(\frac{u\pi(H-1)}{H}\right)\sin\left(\frac{u\pi k}{H}\right)}{\left(1-2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right)^2} \\
&\quad + \frac{2}{H}\left(\sqrt{\frac{q}{p}}\right)^k \cdot \sum_{j=1}^{H-1} (c_j + w_j) \cdot \left(\sqrt{\frac{p}{q}}\right)^j \cdot \sum_{u=1}^{H} \frac{\sin\left(\frac{u\pi j}{H}\right)\sin\left(\frac{u\pi k}{H}\right)}{\left(1-2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right)^2}. \qquad (5.7)
\end{aligned}$$

The preceding calculation assumes that $p+q+c_i+w_i = 1$ for $i = 1, 2, \ldots, H-1$. In this case, $c_j + w_j$ in equation (5.7) may be replaced by $(1-p-q)$. This makes sense as, for the sake of computing the expected duration of the game, the states 0 and H may be combined into a single state.

Note that if $p+q+c_i+w_i < 1$ for some i then the conditional expected duration of play assuming play comes to an end is $\frac{E[D_k]}{r_k+s_k}$ given by equations (5.4), (5.5), and (5.7).

In Table 1, we illustrate the expected duration of play with $H = 16$ for various pairs of p, q values, assuming again that $p+q+c_i+w_i = 1$ for $i = 1, 2, \ldots, 15$. For specific values of p and q, the expected duration time does not depend on the individual values of c_i and w_i. However, the expected duration is sensitive to changes in the combined probability of $p+q$.

Table 1. Expected duration of play D_k with $H = 16$ for several pairs of p, q values with $p+q+c_i+w_i = 1$ for $i = 1, 2, \ldots, 15$. For fixed p and q, the expected duration time does not depend on the individual values of c_i and w_i

k	p=0.5, q=0.5	p=0.45, q=0.45	p=0.3, q=0.6	p=0.2, q=0.4
0	0	0	0	0
1	15	3.73	2.152	1.4039
2	28	6.06	3.842	2.0194
3	39	7.52	5.167	2.2893
4	48	8.42	6.207	2.4076
5	55	8.97	7.024	2.4595
6	60	9.30	7.664	2.4822
7	63	9.47	8.165	2.4922
8	64	9.52	8.556	2.4966
9	63	9.47	8.857	2.4984
10	60	9.30	9.079	2.4991
11	55	8.97	9.214	2.4984
12	48	8.42	9.222	2.4941
13	39	7.52	8.980	2.4736
14	28	6.06	8.156	2.3798
15	15	3.73	5.894	1.9519
16	0	0	0	0

6. Infinite-time ruin probabilities with all probabilities variable

Assume now that all probabilities are state-dependent as in Figure 7, with $p_i + q_i + c_i + w_i = 1$, for $i = 1, 2, \ldots, H-1$. The system of difference equations in (2.1) now looks like

$$
\begin{aligned}
r_0 &= 1 \\
r_1 &= c_1 + q_1 r_0 + p_1 r_2 \\
r_2 &= c_2 + q_2 r_1 + p_2 r_3 \\
&\vdots \\
r_k &= c_k + q_k r_{k-1} + p_k r_{k+1} \\
&\vdots \\
r_{H-1} &= c_{H-1} + q_{H-1} r_{H-2} + p_{H-1} r_H \\
r_H &= 0.
\end{aligned}
\tag{6.1}
$$

Since c_1 and q_1 are both probabilities from state 1 to state 0, we will without loss of generality set $q_1 = 0$. That is, in what follows one may replace c_1 with $c_1 + q_1$ if one wishes to exhibit explicitly both these probabilities. Solving for r_k, $k = 2, 3, \ldots, H-1$, in terms of r_1, the first few results are

$$
\begin{aligned}
r_2 &= \frac{1}{p_1}\left[r_1 - c_1\right] \\
r_3 &= \frac{1}{p_1 p_2}\left[(1 - p_1 q_2)\, r_1 - (c_1 + p_1 c_2)\right]
\end{aligned}
$$

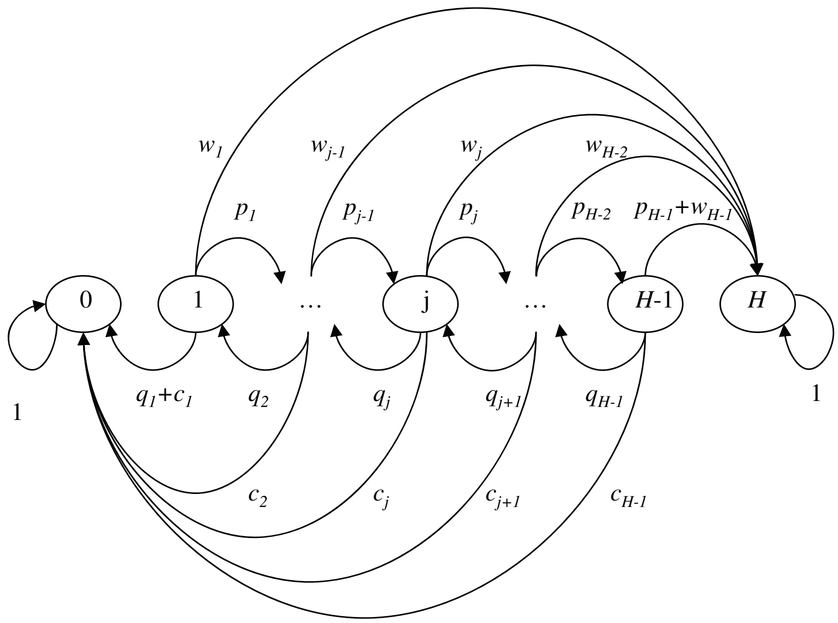

Figure 7. Transition state diagram of Gambler's Ruin with all probabilities state-dependent

$$
\begin{aligned}
r_4 &= \frac{1}{p_1p_2p_3}\left[(1-p_1q_2-p_2q_3)\,r_1-\left[(1-p_2q_3)\,c_1+p_1c_2+p_1p_2c_3\right]\right]\\
r_5 &= \frac{1}{p_1p_2p_3p_4}\left[(1-p_1q_2-p_2q_3-p_3q_4+p_1q_2p_3q_4)\,r_1\right.\\
&\quad \left.-\left[(1-p_2q_3-p_3q_4)\,c_1+(1-p_3q_4)\,p_1c_2+p_1p_2c_3+p_1p_2p_3c_4\right]\right].
\end{aligned}
$$

In order to describe the emerging pattern more compactly, we introduce new notation. Given the sequences $\{p_n\}$ and $\{q_n\}$ describing the state-dependent win and loss probabilities, define $T_k(j)$ as follows:

$$
\begin{aligned}
T_1(1) &= 1\\
T_2(1) &= 1\\
T_3(1) &= 1-p_1q_2\\
T_4(1) &= 1-p_1q_2-p_2q_3\\
T_5(1) &= 1-p_1q_2-p_2q_3-p_3q_4+p_1q_2p_3q_4.\\
&\vdots
\end{aligned}
$$

In general,

$$
T_{k+2}(1)=T_{k+1}(1)-p_kq_{k+1}T_k(1). \tag{6.2}
$$

The argument j of $T_k(j)$ will describe the starting subscript in the win probabilities appearing in $T_k(j)$, $k > 2$. That is, for $k = 3$,

$$\begin{aligned} T_3(1) &= 1 - p_1 q_2 \\ T_3(2) &= 1 - p_2 q_3 \\ T_3(3) &= 1 - p_3 q_4 \\ &\vdots \\ T_3(j) &= 1 - p_j q_{j+1}. \end{aligned}$$

The recursion relation (6.2) becomes

$$T_{k+2}(j) = T_{k+1}(j) - p_{k+j-1} q_{k+j} T_k(j).$$

with

$$T_1(j) = T_2(j) = 1, \qquad \text{for all } j > 0.$$

With this notation,

$$r_k = \frac{1}{p_1 p_2 \cdots p_{k-1}} \left[T_k(1) r_1 - \sum_{i=1}^{k-1} T_i(k-i+1) p_0 p_1 p_2 \cdots p_{k-(i+1)} c_{k-i} \right] \quad (6.3)$$

follows inductively where we define $p_0 = 1$ for notational convenience.

From (6.1), using $r_H = 0$,

$$r_{H-1} = c_{H-1} + q_{H-1} r_{H-2}. \quad (6.4)$$

Using (6.3) to substitute into (6.4) for r_{H-1} and r_{H-2}, and solving for r_1, we obtain

$$\begin{aligned} r_1 &= \frac{\left[\sum_{i=1}^{H-2} T_i(H-i) p_0 p_1 \cdots p_{H-i-2} c_{H-i-1} \right] + p_1 p_2 \cdots p_{H-2} c_{H-1}}{T_{H-1}(1) - p_{H-2} q_{H-1} T_{H-2}(1)} \\ &\quad - \frac{\left[q_{H-1} p_{H-2} \sum_{i=1}^{H-3} T_i(H-i-1) p_0 p_1 \cdots p_{H-i-3} c_{H-i-2} \right]}{T_{H-1}(1) - p_{H-2} q_{H-1} T_{H-2}(1)} \qquad (6.5) \\ &= \frac{\sum_{i=1}^{H-3} [T_{i+1}(H-i-1) - p_{H-2} q_{H-1} T_i(H-i-1)] \cdot [p_0 p_1 \cdots p_{H-i-3} c_{H-i-2}]}{T_H(1)} \\ &\quad + \frac{p_1 p_2 \cdots p_{H-3} [p_{H-2} c_{H-1} + c_{H-2}]}{T_H(1)} \qquad (6.6) \end{aligned}$$

where we again recall that for notational convenience we have combined the effects of c_1 and q_1 by assuming $q_1 = 0$; c_1 can be replaced by $c_1 + q_1$ in order to keep the two effects separate. Substituting this formula into (6.3) gives an expression for r_k, $k = 1, 2, \ldots, H-1$.

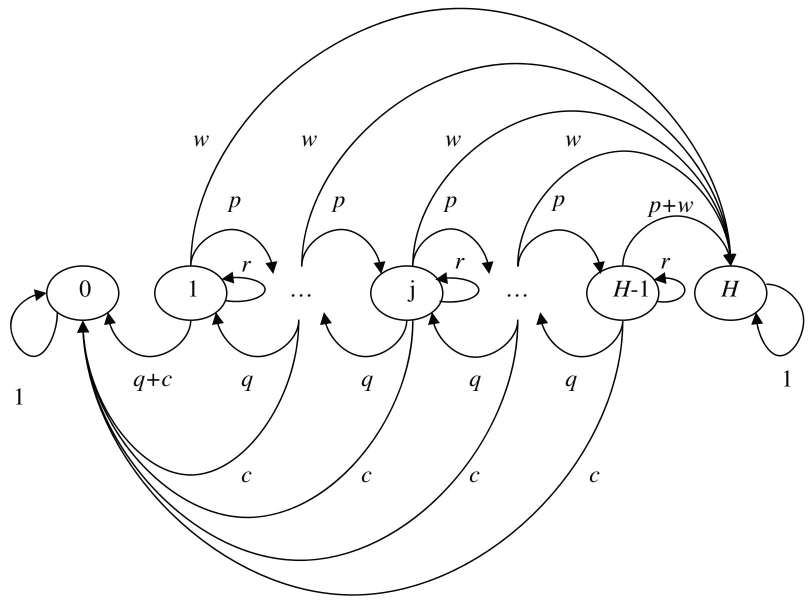

Figure 8. Transition state diagram of Gambler's Ruin with constant catastrophe, windfall, and return probabilities.

For completeness, as in Section 4, the following explicit expression for $T_a(b),\ \ a,b = 1$, 2, 3, …, is given:

$$\begin{aligned} T_a(b) \;=\; & 1 - \sum_{i=b}^{a+b-3} p_i q_{i+1} + \sum_{i=b}^{a+b-5} p_i q_{i+1} \left(\sum_{j=i}^{a+b-5} p_{j+2} q_{j+3} \right) \\ & - \sum_{i=b}^{a+b-7} p_i q_{i+1} \left(\sum_{j=i}^{a+b-7} p_{j+2} q_{j+3} \left[\sum_{k=j}^{a+b-7} p_{k+4} q_{k+5} \right] \right) + \ldots \end{aligned}$$

with the convention that summations are set equal to zero when the upper limit of the summation is less than the lower limit.

7. Related problems and future work

A natural follow-up question is: how would the preceding ruin probabilities change if the transition diagram included return probabilities? The simplest example of this may be seen by altering the model described in Figure 2 into the model shown in Figure 8, where $p+q+r+c+w = 1$. The reader is welcome to verify that the difference equation approach that led to equation (2.2) in Section 2 still applies. This time the roots of the characteristic polynomial and the coefficients have changed but the particular solution remains the same as before.

More generally, the recursive algorithm method developed within Section 6 still produces expressions for the ruin probabilities when applied to the transition diagram in Figure 9,

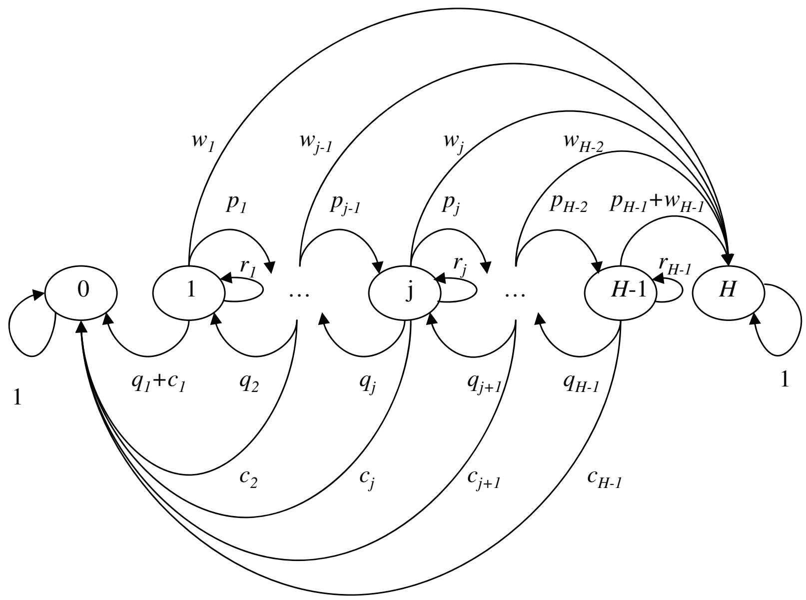

Figure 9. Transition state diagram of Gambler's Ruin with state-dependent win, loss, catastrophe, windfall and return probabilities.

where $p_i + q_i + r_i + c_i + w_i = 1$ for $i = 1, 2, \dots H-1$ with equation (6.2) replaced by

$$T_{k+2}(1) = (1 - r_{k+1})\, T_{k+1}(1) - p_k q_{k+1} T_k(1). \tag{7.1}$$

The remaining details are left to the reader.

Including constant return probabilities in the finite-time models of Sections 3 and 5 would require a lattice path combinatoric analysis which we do not explore here.

The finite-time ruin probability of equation (5.4) suggests that the chance of being ruined by a catastrophe may be analyzed separately. More specifically, r_k is seen to be the sum of two terms. The first term represents the probability that ruin occurs by means of a classical ruin path, *i.e.*, a path having 1 up or 1 down transitions only. The second term is the probability that ruin will occur by way of a catastrophe. Questions concerning the size of the contributions of these terms or conditional probabilities (or expectations) of catastrophes assuming eventual ruin may now be studied further.

Our expressions from Section 5 concerning the expected time to ruin or the expected duration of play may next be analyzed when p and q are variable (or state dependent). Lattice path combinatoric arguments will be needed to develop results in this direction.

Finally, we note that the recursive approach to find infinite-time ruin probabilities presented in Sections 4 and 6 generalizes to other transition diagrams. For example, an expression for the (infinite-time) ruin probability of a system having 1 step up or 2 steps down transitions has been derived and will appear elsewhere.

8. Conclusion

The main results in this article consist of recursive approaches to compute ruin probabilities in a variety of Gambler's Ruin problems which have been generalized to include catastrophe and windfall probabilities as well as the traditional win and loss probabilities. Ruin probabilities of these systems, in both infinite-time and finite-time, have been obtained using different methods of solution including difference equations, lattice path combinatorics and pattern recognition. Numerical examples illustrating these ruin probabilities are presented. Solutions to questions concerning the expected duration of play and the expected time to ruin (conditioned upon the assumption that ruin will occur) have been developed along the way. Some of the recursive methods are robust and provide ruin probability solutions for even more generalized types of transition diagrams.

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