Math 137/331 Real Analysis I
HW 6

1.

a) Evaluate $\int\int xy\, e^{-(x^2+y^2)}\; dxdy$ for $x \geq 0$ and $0 \leq y \leq 1$.

b) Show that $\displaystyle\int_0^\infty \frac{e^{-x} - e^{-2x}}{x}\; dx = log 2.$

Hint for b) integrate e^{-xy} and use Fubini's Theorem.

■

2. Show that for each measurable set E in $\mathbb{R}$ the set

$$\sigma(E) : \{(x, y) : \quad x - y \in E\}$$

is a measurable subset of $\mathbb{R}^2$.

Hint: consider the cases when E is open, E is G_δ, E has measure zero, and E is measurable.

■

3. Consider the set of positive integers with the counting measure. State the Fubini's and Tonelli's theorems for this case.

■

4. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1 & \text{if x} \geq 0 \text{ and } x \leq y < x+1 \\ -1 & \text{if x} \geq 0 \text{ and } x+1 \leq y < x+2 \\ 0 & \text{otherwise} \end{cases}$$

Show that the iterated integrals are not equal. Why does this not contradict the Fubini's theorem?

■

5. Show that $\displaystyle\int_0^\infty x^{2n} e^{-x^2} dx = \frac{(2n)!}{2^{2n} n!} \cdot \frac{\sqrt{\pi}}{2}$ holds true for $n = 0, 1, 2, \cdots$

Hint: Use induction on n and the fact that $\displaystyle\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ (Euler's formula)

■

6. Let $f, g, h \in L^1(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$, and

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)dy.$$

show that

a) $(\alpha f + \beta g) * h = \alpha(f \ * h) + \beta(g * h)$

b) $f * g = g * f$

c) $(f * g) * h = f * (g * h)$

■

7. Let $c \in (0, \infty)$ and $c = m(B(0,1))$, where by $B(x,r)$ we mean the open ball centered at x radius r and $S(x,r)$ is the sphere. Prove that for any $x \in \mathbb{R}^n$ and any $r \in (0, \infty)$, we have

a) $m(B(x,r)) = r^n c$

b) $m(\overline{B(x,r)}) = r^n c$

c) m(S(x,r))=0

d) For fixed $x_0 \in \mathbb{R}^n$ and $r_0 \in (0, \infty)$, we have:

$$\lim_{x \to x_0} \chi_{B(x,r_0)}(y) = \chi_{B(x_0,r_0)}(y)$$

for $y \notin S(x_0, r_0)$. Thus $\lim\limits_{x \to x_0} \chi_{B(x,r_0)} = \chi_{B(x_0,r_0)}$ a.e. in $\mathbb{R}^n$.

e) For fixed $r_0 \in (0, \infty)$ and $x_0 \in \mathbb{R}^n$, we have:

$$\lim_{r \to r_0} \chi_{B(x_0,r)}(y) = \chi_{B(x_0,r_0)}(y)$$

for $y \notin S(x_0, r_0)$. Thus $\lim\limits_{r \to r_0} \chi_{B(x_0,r)} = \chi_{B(x_0,r_0)}$ a.e. in $\mathbb{R}^n$.

■