Math 137/331 Real Analysis I
HW 3

1. Show that the set of irrational numbers is not a countable union of closed subsets of $\mathbb{R}$.

Hint: Use Baire's theorem

Theorem(Baire):If (X, d) is a complete metric space and $X = \bigcup_{n=1}^{\infty} A_n$, then $(\overline{A_n})^{\circ} \neq \emptyset$ for some n. (Ref: A Problem Book in Real Analysis- Aksoy, Khamsi-Page 204)

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2.

a) Show that a set is F_σ if and only if its complement is a G_δ.

b) Consider a real-valued function $f : X \to \mathbb{R}$. The **oscillation** $\omega_f(x)$ of f at the point x is the non-negative extended real number defined by

$$\omega_f(x) = \inf_{V \in \mathfrak{N}_x} \{ \sup_{z,y \in V} |f(z) - f(y)| \}.$$

Where $\mathfrak{N}_x$ denotes the collection of all neighborhoods of the point x.
Show that f is continuous at x if and only if $\omega_f(x) = 0$.

c) Let D denotes the set of all discontinuity of f, i.e., $D = \bigcup_{n=1}^{\infty} D_n$ where $D_n = \{x \in X : \ \omega_f(x) \geq \frac{1}{n}\}$. Show that the set D of all points of discontinuity of f is an F_σ-set. In particular, the set of points of continuity of f is a G_δ-set.

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3. Let ϕ be continuous on $\mathbb{R}$ and let f be finite a.e. in $E \subset \mathbb{R}^d$, so that in particular $\phi \circ f$ defined a.e. in E.

(a) Show that $\phi(f)$ is measurable if f is.

(b) Show that $|f|, |f|^p (p > 0), e^{cf}$ are measurable if f is.

(c) Give an example of a function f which is not measurable but $|f|$ is measurable.

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4. Let A be a dense subset of $\mathbb{R}$. Show that f is measurable if $\{x : \ f(x) > a\}$ is a measurable set for all $a \in A$.

5. Let $f : [0,1] \to [0,1]$ be the Cantor function and let $g(x) = f(x) + x$. Show that

a) g is a bijection from $[0,1]$ to $[0,2]$ and that $h = g^{-1}$ continuous from $[0,2]$ to $[0,1]$.

b) $m(g(C)) = 1$ where C is the Cantor set.

c) Use Problem 5 in HW 2 to deduce that $g(C)$ contains a nonmeasurable set A. Let $B = g^{-1}(A)$ Show that B is measurable (or Lebesgue measurable) but not Borel.

d) Show that there exists a Lebesgue measurable function F and a continuous function G on $\mathbb{R}$ such that $F \circ G$ is not Lebesgue measurable.

Note: Let $C = \{x : \ x = \sum_{j=1}^{\infty} \frac{a_j}{3^j} \ \ a_j = 0, 2 \text{ for all} j\}$. Let $f(x) = \sum_{j=1}^{\infty} \frac{b_j}{2^j}$ where $b_j = \frac{a_j}{2}$. The series defining $f(x)$ is the base 2 expansion of a number in $[0,1]$, and any number in $[0,1]$ can be obtained in this way. Hence f maps C onto $[0,1]$. Note that if $x, y \in C$ and $x < y$ then $f(x) < f(y)$ unless x and y are the end points of one of the intervals removed from $[0,1]$ to obtain C. In this case $f(x) = \frac{p}{2^k}$ for some integers p, k, and $f(x)$ and $f(y)$ are two base-2 expansions of this number. Extend f from $[0,1]$ to itself by declaring it to be constant on each interval missing from C. This extended f is still increasing, and since its range is all of $[0,1]$ it cannot have any jump discontinuities; hence it is continuous. f is called the **Cantor function** or Cantor-Lebesgue function.

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6. Show that

a) The sum and product of two simple functions are simple.

b) $\chi_{A\cap B} = \chi_A \cdot \chi_B$

c) $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$

d) $\chi_{A^c} = 1 - \chi_A$

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