xxx
Real Analysis I
HW 2
Due 09/??/2013

1.Given a subset E of $\mathbb{R}$, define exterior measure of E by

$$m_*(E) = \inf\{\sum_{n=1}^{\infty} l(I_n) : \quad E = \bigcup_{n=1}^{\infty} I_n\}$$

where by $l(I_n)$ we mean the length of I_n and the infimum is taken over all coverings of E by countable bounded intervals. Prove the following:

a) $0 \leq m_*(E) \leq \infty$

b) If $E \subset F$ then $m_*(E) \leq m_*(F)$

c) $m_*(E + x) = m_*(E)$ where $E + x = \{e + x : \; e \in E\}$

d) $m_*(E) = 0$ for any countable set E.

e) $m_*(E) < \infty$ for any bounded set E

f) $m_*(E) = \inf\{\sum_{n=1}^{\infty}(b_n - a_n) : \quad E = \bigcup_{n=1}^{\infty (a_n, b_n)}$.

■

2. Let $E = \mathbb{Q} \cap [0, 1]$, and $\{I_n\}$ be a **finite** collection of open intervals covering E. Show that $\sum_{n=1}^{\infty} |I_n| \geq 1$. What can you conclude from this question?

■

3. Stein and Shakarchi text: page 39 numbers: 5,6,7,8

■

4. Prove the **The Borel-Cantelli lemma**.
See Stein and Shakarchi text: page 42 number: 16.
Remark:In Probability theory the Borel-Cantelli lemma is stated as follows: Given $\{E_n\}$ sequence of events in some probability space, if the sum of the probabilities of E_n is finite i.e., if $\sum_{n=1}^{\infty} m(E_n) < \infty$ then the probability that infinitely many of them occur is 0. i.e., $m(\text{limsup} E_n) = 0$.

■