Math 138/332 Real Analysis II
HW 3-Due March 1st, 2011

1. Let M be a closed subspace of a Hilbert space H and $x_0 \in H$. Prove:

$$d(x_0, M) = \sup\Big\{|(x_0, y)| : \ \ y \in M^{\perp}, \ \ ||y|| = 1\Big\}$$

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2.

a) Show that if M and N are closed, orthogonal subspaces of a Hilbert space, then $M + N$ is closed.

b) If M is finite dimensional N is a closed (but not orthogonal to M) subspaces of H, then show that $M + N$ is closed.
Hint: Prove $M+N$ is complete. There is no loss of generality in assuming $\dim M = 1$.

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3.

a) Let H be a Hilbert space, $a, b \in H$, $a \neq 0, b \neq 0$ are two orthogonal elements and $U : H \to H$ is defined by
$$U(x) = a(x, b) + b(x, a).$$
Calculate $||U||$

b) Using $a)$ calculate $||U||$, where $U : L^2[0, \pi] \to L^2[0, \pi]$ is defined by

$$Uf(x) = sinx \int_0^{\pi} f(t) cost \, dt + cosx \int_0^{\pi} f(t) sint \, dt$$

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4. Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{C}$ given by

$$f_n(x) = \pi^{-\frac{1}{2}} \frac{(x - i)^n}{(x + i)^{n+1}}$$

Prove that the family $\{f_1, f_2, \cdots\}$ is orthonormal in $L^2(\mathbb{R})$, that is

$$\int_{-\infty}^{\infty} f_m(x) \overline{f_n(x)} \, dx = \begin{cases} 1 & \text{if } \ \text{m} = \text{n} \\ 0 & \text{if } \ \text{m} \neq n \end{cases}$$

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5.

a) Let H be a Hilbert space and $\{x_n\}_{n\in N} \subseteq H$ an orthonormal system. Prove that $x_n \to 0$ weakly.

b) Let $A \subseteq [0, 2\pi]$ be a Lebesgue measurable set. Prove that

$$\lim_{n\to\infty} \int_A sin(nt)\, dt = \lim_{n\to\infty} \int_A cos(nt)\, dt = 0$$

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6.
Let H be a Hilbert space $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis and $x^* : H \to K$ is linear and continuous functional. Prove that $y = \sum_{n=1}^{\infty} \overline{x^*(e_n)} e_n$ is the unique element in H with the property that $x^*(x) = (x, y)$ for all $x \in H$.

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7. Let P be the orthogonal projection associated with a closed subspace S in a Hilbert space H, that is $P(f) = f$ if $f \in S$ and $P(f) = 0$ if $f \in S^{\perp}$

a) Show that $P^2 = P$ and $P^* = P$.

b) Conversely if P is any bounded operator satisfying $P^2 = P$ and $P^* = P$, prove that P is the orthogonal projections for some closed subspace of H.

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8.

a) Let H be a Hilbert space, $n \in \mathbb{N}$, $x_1, x_2, \ldots x_n \in H$ are n linearly independent elements, and $S = \text{span}\{x_1, x_2, \ldots x_n\}$. Prove that for any $x \in H$,

$$P_S(x) = \frac{\overline{\Delta_1}}{\Delta} x_1 + \frac{\overline{\Delta_2}}{\Delta} x_2 + \cdots + \frac{\overline{\Delta_n}}{\Delta} x_n$$

Where Δ is the Gram determinant and Δ_i are obtained by replacing in Δ the ith column with the column

$$\begin{pmatrix} (x_1, x) \\ (x_2, x) \\ \cdots \\ (x_n, x) \end{pmatrix}$$

b) Use part a) to calculate the orthogonal projection of $x = (4, -1, -3, 4)$ onto the linear subspace spanned by $x_1 = (1, 1, 1, 1)$, $x_2 = (1, 2, 2, -1)$, $x_3 = (1, 0, 0, 3)$.