Math 138/332 Real Analysis II
HW 2 - Due February 15, 11

1. Show that the inner product in a Hilbert space is jointly continuous. That is if $x_n \to x$ and $y_n \to y$, then $(x_n, y_n) \to (x, y)$ as $n \to \infty$.

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2. Let $f \in \ell^2$, define

$$|||f||| = ||f||_2 + ||f||_\infty = (\sum_{n=1}^{\infty} |f(n)|^2)^{1/2} + \max|f(n)|.$$

Show that $(\ell^2, |||.|||)$ is not an inner product space.

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3. If $a = \{a_k\}$ belongs to ℓ^p for some $p < \infty$, then show that

$$\lim_{p\to\infty} ||a||_p = ||a||_\infty$$

where $||a||_p = (\sum_k |a_k|^p)^{1/p}$ for $0 < p < \infty$; $\quad ||a||_\infty = \sup_k |a_k|$.
Hint: Since $|a_k| \to 0$, there is a largest $|a_k|$ say $|a_{k_0}|$ such that $||a||_\infty = |a_{k_0}|$.

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4. Consider $L^p(X)$ where (X, μ) is a measure space and $p \in (1, \infty)$. Let $q \in (1, \infty)$ be the conjugate of p. We say a sequence (f_n) in $L^p(X)$ **converges weakly** to an element f in $L^p(X)$ if

$$\lim_{n\to\infty} \int_X f_n g \, d\mu = \int_X fg \, d\mu$$

for every $g \in L^q(X)$. Show that if a sequence $(f_n) \in L^p$ converges to an element of $f \in L^p$ in the norm of L^p, then $(f_n) \in L^p$ converges weakly to f.

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5. Suppose f is a measurable function on $\mathbb{R}^n$

a) Show that f is essentially bounded on $\mathbb{R}^n$ if and only if $||f||_\infty < \infty$

b) Show that $|f| \leq ||f||_\infty$

c) If $||f||_\infty < \infty$, then show that f is essentially bounded and $||f||_\infty$ is an essential bound for f.

Hint for part b): When $||f||_\infty < \infty$, for every $k \in \mathbb{N}$ there exists an essential bound M_k of f such that $M_k < ||f||_\infty + 1/k$, furthermore

$$\{x : \ |f(x)| > ||f||_\infty\} = \bigcup_{k\in\mathbb{N}} \{x : \ |f(x)| > ||f||_\infty + 1/k\}$$

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6. (**Hölder's inequality for $p = 1$ and $q = \infty$**) Suppose f and g are two measurable functions such that $|f|$, $|g| < \infty$ a.e.x

a) Excepting the case that one of $||f||_1$ and $||g||_\infty$ is equal to 0 and the other is equal to ∞ show that

$$||fg||_1 \leq ||f||_1 ||g||_\infty$$

b) When $||f||_1$, $||g||_\infty < \infty$, show that the equality in the above inequality holds if and only if

$$|g| = ||g||_\infty \text{ a.e. on } \ \{x : \ f(x) \neq 0\}$$

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7. Let X be a measure space with $m(X) = 1$.

a) If f and g are in $L^1(X)$ are two positive functions satisfying $f(x) \cdot g(x) \geq 1$ for almost all x, show that

$$\Big(\int f dx\Big)\Big(\int g dx\Big) \geq 1$$

b) If $f, g \in L^2(X)$ with $\displaystyle\int f\, dx = 0$, show that

$$\Big(\int f \cdot g\, dx\Big)^2 \leq \Big[\int g^2\, dx - \Big(\int g\, dx\Big)^2\Big] \cdot \int f^2\, dx$$

Hint: Both parts a) and b) require Hölder's inequality. For part b) set $\alpha = \displaystyle\int g\, dx$ and observe $\displaystyle\Big|\int f \cdot g\, dx\Big| = \Big|\int (f \cdot g - \alpha \cdot f)\, dx\Big|$

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