Math 138/332 Real Analysis II
HW 1-Due February 1st, 2011

1. Let $X \neq 0$ be a real or complex linear space. Prove that there is at least one norm on X.
Hint. Every linear space has a basis.

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2. Given a function $p : X \to [0, \infty)$ with the properties

a) $p(x) = 0 \Leftrightarrow x = 0$

b) $p(\lambda x) = |\lambda| p(x)$ for all $x \in X$ and $\lambda \in K$.

Show that p is a norm if and only if $B_X = \{x \in X : \ p(x) \leq 1\}$ is convex.
(i.e.,The triangle axiom and the convexity of the closed unit ball are equivalent)

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3. Let $a > 0$. On $C[0, 1]$ consider the following norms:

$$||f||_\infty = \sup_{t\in[0,1]} |f(t)|$$

$$||f||_1 = a \int_0^1 |f(t)| dt.$$

Prove that $||f|| = \ \min\{||f||_\infty, ||f||_1\}$ is a norm on $C[0, 1]$ if and only if $a \leq 1$.
This problem shows minimum of two norms is not a norm in general. Show that maximum of two norms is indeed a norm.

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4. Let $X = \mathfrak{M}_{n,m}(\mathbb{R})$ be the real vector space of $n \times m$ matrices with real entries. Given $A, B \in \mathfrak{M}_{n,m}(\mathbb{R})$, set

$$(A, B) = tr(A^t B)$$

where by "tr" we mean the trace of a square matrix. i.e., the sum of the entries lying in the diagonal.

a) Show that $(., .)$ is an inner product on $\mathfrak{M}_{n,m}(\mathbb{R})$.

b) Deduce that $A \mapsto ||A|| = \sqrt{tr(A^t A)}$ is a norm on X.

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5. Show that every Hilbert space is uniformly convex.
A normed linear space is said to be **uniformly convex** if for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ independent of x and y such that $||x|| \leq 1$, $||y|| \leq 1$, $||x - y|| \geq \epsilon$ implies $||\frac{1}{2}(x + y)|| \leq 1 - \delta$.

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6.

a) Suppose f is supported on a set E of finite measure. If $f \in L^2(\mathbb{R}^n)$, then show that $f \in L^1(\mathbb{R}^n)$ and that
$$||f||_{L^1(\mathbb{R}^n)} \leq m(E)^{\frac{1}{2}} ||f||_{L^2(\mathbb{R}^n)}$$

b) If $|f(x)| \leq M$ (f is bounded) and $f \in L^1(\mathbb{R}^n)$, then $f \in L^2(\mathbb{R}^n)$ and that
$$||f||_{L^2(\mathbb{R}^n)} \leq M^{\frac{1}{2}} ||f||_{L^1(\mathbb{R}^n)}^{\frac{1}{2}}$$

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7. Show that $L^2(\mathbb{R}^n)$ is **separable**. i.e., There exists a countable dense set in $L^2(\mathbb{R}^n)$.

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