Your Name
Math Analysis II
HW 7
04/09/13

1. Let $f(x,y) = (\cos y + x^2, e^{x+y})$ and $g(u,v) = (e^{u^2}, u - \sin v)$
Write a formula for $f \circ g$ and calculate $D(f \circ g)(0,0)$ using the chain rule.

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2. If $f(0,0) = 0$ and

$$f(x,y) = \frac{xy}{x^2+y^2} \text{ if } (x,y) \neq (0,0)$$

Prove that partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exists at every point of $\mathbb{R}^2$, although $f$ is not continuous at $(0,0)$.

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3. Suppose $f$ is a differentiable mapping of $\mathbb{R}$ into $\mathbb{R}^3$ such that $|f(t)| = 1$ for every $t$. Prove that $f'(t) \cdot f(t) = 0$. Interpret this result geometrically.

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4. We say $f : \mathbb{R}^n \to \mathbb{R}$ is **homogeneous of degree** $l$ if for all $\lambda > 0$ $f(\lambda p) = \lambda^l f(p)$ holds.
For example the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined as $f(x,y,z) = x^2 + 2yz$ is homogenous of degree 2.
Prove that $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree $l$ if and only if $p \cdot \triangledown f(p) = l f(p)$
Hint: Consider a new function defined as $F(x_1, x_2, \cdots x_n, \lambda) = \lambda^{-l} f(\lambda x_1, \lambda x_2, \cdots, \lambda x_n)$

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5. Use Hessian Criterion to identify the local extrema for the following functions:

a $f(x,y) = \ln(x^2 + y^2 + 1)$

b $f(x,y,z) = x^3 + xz^2 - 3x^2 + y^2 + 2z^2$

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6. Prove that for every $A \in L(\mathbb{R}^n, \mathbb{R})$ corresponds to a unique $y \in \mathbb{R}^n$ such that $Ax = x \cdot y$.
Prove also $||A|| = |y|$.

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7. Suppose that $f$ is a differentiable mapping of a connected open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, and if $f'(x) = 0$ for every $x \in \mathbb{R}^n$, prove that $f$ is constant in $E$.

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8. Use the one dimensional MVT to prove: If $f, f_x$ and $f_y$ are continuous on a circular region containing $A(x_0, y_0)$ and $B(x_1, y_1)$ then there is a point $(x^*, y^*)$ on the line segment joining the points $A$ and $B$ such that

$$f(x_1, y_1) - f(x_0, y_0) = f_x(x^*, y^*)(x_1 - x_0) + f_y(x^*, y^*)(y_1 - y_0)$$

This result is known as two dimensional MVT.

[Hint: Set $F(t) = f(x(t), y(t))$ where $x(t)$ and $y(t)$ represents the parametric equation of the line connecting $A$ to $B$, then apply the one dimensional MVT to $F(t)$ on the interval $[0, 1]$.]

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9. Let $f$ be continuous on $[a, b] \times [c, d]$ and; for $a < x < b$, $c < y < d$, define

$$F(x, y) = \int_a^x \int_c^y f(u, v) dv du.$$

Show that

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} = f(x, y)$$

Use this example to discuss the relationship between Fubini's Theorem and equality of mixed partial derivatives.

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