Your Name
Math Analysis II
HW 5
03/7/2013

1.

a) The map $B_n : \mathcal{C}[0,1] \to R$ defined by

$$B_n(f)(x) = \sum_{k=0}^{n} f(\frac{k}{n}) \left(\begin{array}{c} n \\ k \end{array} \right) x^k (1-x)^{n-k}$$

is linear and monotone.

b) Show that $B_n 1 = 1$ and $B_n x = x$

■

2.

a) Show that n-th Bernstein polynomial for $f(x) = e^x$ is $B_n(x) = [1 + (e^{\frac{1}{n}} - 1)x]^n$

b) Show that this may be rewritten as $(1 + \frac{x}{n} + \frac{c_n}{n^2})^n$ where $0 \leq c_n \leq 1$.

c) Hence prove directly that $B_n(e^x)$ converges uniformly to e^x on $[0,1]$.

■

3.
Prove that if f is a continuous function on the closed and bounded interval $[a,b]$, then for any $\epsilon > 0$, there is a piecewise linear function F which approximates f uniformly within ϵ on the interval.

Hint: Divide the interval $[a,b]$ into N equal subintervals at points

$$a = x_0 < x_1 < x_2 < ... < x_N = b$$

Let P_k be a the point (x_k, y_k) where $y_k = f(x_k)$ and define the function F on each subintervals by

$$F(x) = \frac{(x_{k+1} - x)y_k + (x - x_k)y_{k+1}}{x_{k+1} - x_k}$$

■

4.

a) Show that for any function $f \in C[0,1]$ and any number $\epsilon > 0$, there exist a polynomial p, all of whose coefficients are rational numbers, such that

$$||p - f|| < \epsilon.$$

b) Show that $C[a,b]$ is separable.

■

5.

a) Show that if $f \in C[a,b]$, and if $\int_a^b x^n f(x)dx = 0$ for each $n = 1, 2, ...$ then $f = 0$.

b) For a $f \in C[a,b]$, the moments of f are the numbers

$$\mu_n = \int_a^b x^n f(x)dx$$

where $n = 0, 1, 2, ...$ Prove that two continuous functions defined on $[a,b]$ are identical if they have the same sequence of moments.

■

6.

a) Show that the function $x \mapsto e^x$ on $\mathbb{R}$ is not the uniform limit on $\mathbb{R}$ of a sequence of polynomials. Hence the Weierstrass Approximation Theorem may fail for infinite integrals.

b) Show that the Weierstrass Approximation Theorem fails for bounded open intervals.

■

7. Suppose a, b, c and d are constants chosen from an interval $[-K, K]$ and let $\Phi \subset (C[0,\pi], d)$ be a family of functions f of the form

$$f(x) = a\sin bx + c\cos dx \quad \text{where} \quad 0 \leq x \leq \pi.$$

Where the metric d on $C[0,\pi]$ is the uniform metric $d(f,g) = \max_{0\leq x\leq \pi}|f(x) - g(x)|$.

a) Show that Φ is a compact subset of $(C[0,\pi], d)$

b) Show that for any continuous function g defined on $(C[0,\pi], d)$ there exists a, b, c and d in $[-K, K]$ such that

$$\max_{0\leq x\leq \pi}|g(x) - (a\sin bx + c\cos dx)|$$

is minimum. For obvious reasons $f \in \Phi$ is called minimax approximation of of g.

■