Math Analysis I
HW 8
Due 04/12/2013

1. Which of the following sets are compact?

a) $[0,1] \cup [5,6] \subset \mathbb{R}$

b) $\{x \in \mathbb{R} : x \geq 0\} \subset \mathbb{R}$

c) $\{x \in \mathbb{R} : 0 \leq x \leq 1$ and x is irrational$\}$

d) $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\} \cup \{0\}$

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2. Prove that a closed subset of a compact set is compact.

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3.

a) Let $r_1, r_2, r_3, \cdots$ be an enumeration of the rational numbers in the interval $[0,1]$. Show that there is a convergent subsequence.

b) Let A be a bounded set in $\mathbb{R}^n$. Prove that $\overline{A}$ is compact.

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4. Let $A \subset \mathbb{R}^n$ be compact and let x_k be a Cauchy sequence in $\mathbb{R}^n$ with $x_k \in A$. Show that x_k converges to a point in A.

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5. Let$M$ be a complete metric space and F_n a collection of closed nonempty subsets (not necessarily compact) of M such that $F_{n+1} \subset F_n$ and the diameter of F_n,

$$\delta(F_n) = \sup\{d(x,y) : \ x, y \in F_n\} \to 0$$

Show that $\bigcap_{n=1}^{\infty} F_n$ consist of a single point. Compare this result with Cantor's Intersection Theorem.

Hint: Pick $x_n \in F_n$. Show that (x_n) is a Cauchy sequence. Its limit must be in $\overline{F_n}$ for all n. There cannot be two such points since $\delta(F_n) \to 0$.

6. Show that the cube $[a, b] \times [a, b] \times [a, b]$ is a compact subset of $\mathbb{R}^3$.

7.

a) Let $(x_n) \to x$ be a convergent sequence in a metric space and let

$$A = \{x_1, x_2, \cdots\} \cup \{x\}$$

Show that A is compact.

b) Let M be a set with a discrete metric. Show that any infinite subset of M is non-compact. Why does this not contradict the statement in part a)?

8. Verify the nested sets property for $F_k = \{x \in \mathbb{R} : \ x \geq 0, \ 2 \leq x^2 \leq 2 + (1/k)\}$