Math Analysis I
HW 6
Due 03/26/2013

1. The diameter $\delta(A)$ of a nonempty set A in a metric space (X, d) is defined to be

$$\delta(A) = \sup\{d(x, y) : \ x, y \in A\}$$

a) Show that $\delta(A) = 0$ if and only if A consist of a single point

b) Show that $A \subset B$ implies that $\delta(A) \leq \delta(B)$.

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2. Show that the function

$$\| f \| = \max\{| f(x) | : \ x \in [a, b]\}$$

defines a norm on $C[a, b]$.

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3.

a) Show that $| \| x \| - \| y \| | \leq \| x - y \|$

b) Show that norm is a continuous function.
i.e., show that $x_n \to x \Rightarrow \| x_n \| \to \| x \|$.

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4. Let $\{I_n\}$ be a sequence of bounded non-empty closed subsets of a complete metric space (X, d) such that

a) $I_{n+1} \subseteq I_n$, for all $n \geq 1$;

b) $\lim_{n\to\infty} \delta(I_n) = 0$, where $\delta(A) = \sup\{d(x, y) : x, y \in A\}$.

Show that $\bigcap_{n\geq 1} I_n$ is not empty and reduced to a single point.

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5. Show that a subset $A \subseteq \mathbb{R}$ is open if and only if A is the union of a countable collection of open intervals.

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6. Prove that

a) If A is open and B is closed, then $A\backslash B$ is open and $B\backslash A$ is closed.

b) Let A be open and let B be an arbitrary subset of $\mathbb{R}$. Is AB necessarily open? (Where $AB = \{xy \in \mathbb{R} : x \in A \text{ and } y \in B\}$.)

c) Let $A = \{x \in \mathbb{R} : x \text{ is irrational}\}$. Is A closed?

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7.

a) Given $\overrightarrow{u} = (2+i,\ -3i,\ -1-2i)$ and $\overrightarrow{v} = (3,\ 4-i,\ -1-i)$. Find the norm $\| \overrightarrow{u} \|$ and the complex inner product $< \overrightarrow{u}, \overrightarrow{v} >$ and the distance $d(\overrightarrow{u},\ \overrightarrow{v})$.

b) A complex sequence $(z_n) = (x_n + iy_n) \to z$ if and only if the real part $(x_n) \to x$ and the imaginary part $(y_n) \to y$ and $z = x + iy$. Use this fact to decide whether or not following complex sequences converges
i) $(z_n) = (i^n)$
ii) $(z_n) = (\frac{-1}{2})^n + i(1 - \frac{1}{2n})$.
Note that Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ implies $(z_n) = (i^n) = \{e^{i\frac{\pi}{2}}\}^n = e^{i\frac{n\pi}{2}}$.

8.

a) Show that any discrete metric space (X, d) is complete.

b) Show that the Euclidean space $(\mathbb{R}^2, \| \,.\, \|_2)$ is complete.
Note that a Cauchy sequence (x_n) where $x_n = (x_1^n, x_2^n)$ in $\mathbb{R}^2$ satisfy the condition:
Given $\varepsilon > 0$ there exist an $N \in \mathbb{N}$

$$\| x_k - x_l \|_2 = \sqrt{\sum_{j=1}^{2} |x_j^k - x_j^l|^2} \leq \varepsilon \quad \text{for all} \quad k, l > N$$