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The Radius of the Essential Spectrum

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In this paper we define an operator measure s on $L(X)$ into $R^+ \cup \{0\}$ satisfying suitable conditions. Then letting $I = s^{-1}(0)$, we consider the quotient algebra $L(X)/I$, instead of Calkin algebra and define $\sigma_I(T) = \{\lambda \in \mathbb{C}: \Pi_2(\lambda - T)$ is not invertible in $L(X)/I\}$, where $\Pi_2: L(X) \to L(X)/I$ is the natural homomorphism, and $r_I(T) = \sup\{|\lambda|: \lambda \in \sigma_I(T)\}$. After proving the fact that $\sigma_I(T)$ is equal to the essential spectrum of T and replacing standard measure of noncompactness with suitably defined s-measures we obtain that the radius $r_e(T)$ of the essential spectrum is equal to $\lim_n(s(T^n))^{1/n}$. We also construct examples of such operator measures.

I. THE ESSENTIAL SPECTRUM AND ITS RADIUS

DEFINITION I.1. Let X be a Banach space, $L(X)$ be the bounded linear operators on X and let $\mathscr{K}$ be the closed ideal of compact operators on X. The quotient algebra $L(X)/\mathscr{K}$ is a Banach algebra called the Calkin algebra.

The essential spectrum $\sigma_e(T)$ of T is defined by:

$$\sigma_e(T) = \{\lambda \in C: \Pi_1(\lambda - T) \text{ is not invertible in } L(X)/\mathscr{K}\}$$

where Π_1 is the natural homomorphism from $L(X)$ onto $L(X)/\mathscr{K}$. An operator $T \in L(X)$ is called a Fredholm operator if (i) the range of T is closed and (ii) the kernel of T and the cokernel of T are of finite dimension. For such an operator T, the index $i(T)$ is defined by

$$i(T) = \dim(\ker T) - \dim(\text{cocker } T).$$

The connection between the class of Fredholm operators and the Calkin algebra is contained in the following theorem of Atkinson [3].

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THEOREM I.2. *For $T \in L(X)$ the following are equivalent:*

(1) *T is Fredholm;*

(2) *$\exists H \in L(X)$ such that $1 - TH$ and $1 - HT$ are of finite rank;*

(3) *$\exists H \in L(X)$ such that $1 - TH$ and $1 - HT$ are compact;*

where 1 denotes the identity operator on X.

Therefore an equivalent definition for the essential spectrum is:

$$\sigma_e(T) = \{\lambda \in C: \lambda - T \text{ is not Fredholm}\}.$$

The radius $r_e(T)$ of the essential spectrum $\sigma_e(T)$ is defined by

$$r_e(T) = \sup\{|\lambda|: \lambda \in \sigma_e(T)\}.$$

In this section, we shall consider bounded T and shall obtain some other equivalents for the spectral radius. Our discussion is motivated by the results of Nussbaum [10] and Lebow and Schechter [8]. Nussbaum uses the concept of the ball measure of noncompactness to obtain a formula for $r_e(T)$. For a bounded set D in X, the ball measure of noncompactness of D, denoted by $\gamma(D)$, is defined as

$$\gamma(D) = \inf\left\{r > 0: D \subset \bigcup_{i=1}^{k} B(x_i, r)\right\}.$$

Here $B(x_i, r)$ stands for the ball centered at $x_i \in X$ with radius r and k is arbitrary but finite. We set $\gamma(T) = \gamma(T(\mathcal{U}_X))$. Nussbaum proves that

$$r_e(T) = \lim_n (\gamma(T^n))^{1/n}.$$

Lebow and Schechter [8] show

$$r_e(T) = \lim_n (\|T^n\|_M)^{1/n}, \qquad \text{where} \quad \|T\|_M = \inf\{\eta: \exists a$$

subspace M of finite codimension $\ni \|Tx\| < \eta \, \|x\|$, $x \in M\}$. It can be shown that [1] $\gamma(T) = \lim_n \delta_n(T)$, where $(\delta_n(T))$ are the standard Kolmogorov diameters of T. Also it is easily seen that $\|T\|_M = \lim_n c_n(T)$, where $(c_n(T))$ are the standard Gelfand numbers of T [11]. We set $c(T) = \lim_n c_n(T)$.

DEFINITION I.3. An operator measure s is a map from $L(X)$ into $R^+ \cup \{0\}$ with the following properties:

(0) $s(T) = 0$ if T is of rank 1:

(1) $s(T) \leqslant \|T\|$;

(2) $s(T+H) \leqslant s(T)+s(H)$;

(3) $s(TH) \leqslant s(T)\, s(H)$;

(4) $s(T)=0 \Rightarrow T$ is compact.

Let $s(\cdot)$ be an operator measure and let I be the ideal defined as

$$I=\{T \in L(X): s(T)=0\}.$$

Let $(T_n) \subset I$ and let $\lim_n \|T^n - T\|=0$. Then $s(T)=s(T-T_n+T_n) \leqslant s(T-T_n)+s(T_n) \to 0$. Therefore I is closed and clearly $\mathscr{F} \subset I$, where $\mathscr{F}$ denotes the ideal of finite rank operators. From (4) of Definition I.3 we have $I \subset \mathscr{K}$. Note that if $\mathscr{K}=\bar{\mathscr{F}}$, the uniform closure of $\mathscr{F}$, then $I=\mathscr{K}$. Therefore one can only obtain "nontrivial" ideals I only when $\bar{\mathscr{F}} \subsetneqq \mathscr{K}$. Enflo [6] and Davie [5] have given examples of Banah spaces for which $\bar{\mathscr{F}} \subsetneqq \mathscr{K}$ holds. Now I is closed and therefore $L(X)/I$ is a Banach space, let $\Pi_2: L(X) \to L(X)/I$ be the natural homomorphism. We define the set $\sigma_I(T)$ by

$$\sigma_I(T)=\{\lambda \in \mathbb{C}: \Pi_2(\lambda - T) \text{ is not invertible in } L(X)/I\}.$$

Since $I \subset K$ it is easy to see that $\sigma_e(T) \subset \sigma_I(T)$.

Now the followig proposition shows that the above two sets are the same.

PROPOSITION I.4. *For $T \in L(X)$ we have $\sigma_e(T)=\sigma_I(T)$.*

Proof. We only need to show $\sigma_I(T) \subset \sigma_e(T)$. Now suppose $\lambda \notin \sigma_e(T)$ then $\lambda - T$ is Fredholm, let $T_1=\lambda - T$. Then the well-known facts about Fredholm operators (see Pietsch [11]) give: there exists a Fredholm operator S and finite rank maps F_1 and F_2 such that $ST_1=\mathrm{Id}+F_1$ and $T_1 S=\mathrm{Id}+F_2$. However, $F \subset I$ therefore $\lambda \notin \sigma_I(T)$.

In case $I=\bar{\mathscr{F}}$, he uniform closure of $\mathscr{F}$, the above result has been proved by Mattila [9].

We define $r_I(T)$ by

$$r_I(T)=\sup\{|\lambda|: \lambda \in \sigma_I(T)\}.$$

However, by Proposition I.4 we have $r_I(T)=r_e(T)$. Therefore we would like to replace measures of noncompactness with s-measures to obtain $r_e(T)$ in terms of them and this is achieved in the sequel.

DEFINITION I.5. Let $\sigma_b(T)$ denote Browder's essential spectrum (see Browder [4]) which is the set of all $\lambda \in \sigma(T)$, the spectrum of T, such that at least one of the following conditions holds:

(1) The range of $\lambda - T$ is not closed;

(2) λ is a limit point of $\sigma(T)$;

(3) $[\bigcup_{n \geqslant 1}$ null space $(T-\lambda)^n]$ is infinite dimensional.

There are many other possible definitions of the essential spectrum and, in general, these definitions are not equivalent. Note, however, that Browder's essential spectrum is the largest [8]. Proposition I.6 below follows directly from the well-known work of Gohberg and Krein [7, Theorems 3.3 and 4.2]. Although they have not explicitly defined the essential spectrum, the result stated below and its proof are implicit in Gohberg and Krein. We include a proof for the sake of completeness.

Notation. Let X be a Banach space, $T \in L(X)$, $\delta(T)$ the resolvent set of T,

$$H = \{\lambda \in \mathbb{C} \colon \lambda - T \text{ is Fredholm}\}$$

and

$$H_r = \{\lambda \in \mathbb{C} \colon \lambda - T \text{ is Fredholm and } i(\lambda - T) = r\},$$

where r is a fixed constant

PROPOSITION I.6. *Whenever $H_r \cap \delta(T) \neq \emptyset$ and $\lambda \in H_r$, then $\lambda \in \sigma_b(T)$.*

Proof. Suppose $H_r \cap \delta(T) \neq \emptyset$; let $\lambda_0 \in H_r \cap \delta(T)$ then $T - \lambda_0$ is invertile, therefore $\dim \ker(T-\lambda_0) = \dim \operatorname{coker}(T-\lambda_0) = 0$, which gives $i(T-\lambda_0) = 0$. But $\lambda_0 \in H_r$ and hence $i(T-\lambda) = 0$ for all $\lambda \in H_r$. Now from Theorem 3.3 of Gohberg and Krein [7, p. 205] it follows that $\dim \ker(T-\lambda) = 0$ for all $\lambda \in H_r$ except at possible isolated points $\{\lambda_j\}$. At these isolated points, $T-\lambda$ has closed range, trivial kernel (one-to-one), and trivial cokernel (range dense). Therefore $T-\lambda$ is invertible except for these possible isolated points $\{\lambda_j\}$. This implies $\lambda \notin \sigma_b(T)$ when $T-\lambda$ is invertible. Next, one verifies that $\lambda_j \notin \sigma_b(T)$ by considering the definition of $\sigma_b(T)$; observe for λ_j as above

(1) range of $T-\lambda_j$ is closed since $T-\lambda_j$ is Fredholm,

(2) λ_j is not a limit point of $\sigma(T)$, and

(3) $[\bigcup_{n \geqslant 1}$ null space $(T-\lambda_j)^n]$ is finite dimensional.

(This corresponds to C_{λ_j} of Theorem 4.2 of Gohberg and Krein [7, p. 212] which shows $\lambda_j \notin \sigma_b(T)$.

LEMMA I.7. *Let s be an operator measure and $T \in L(X)$. Let*

$$r_I^* = \inf_n (s(T^n))^{1/n} \qquad \textit{and} \qquad r_I' = \lim_n (s(T^n))^{1/n}.$$

Then

(i) $r_I^* = r'_I$.

(ii) *If* $s(T) \geqslant \gamma(T)$ *and if* $|\lambda| > r'_I$, *then* $\lambda - T$ *is a Fredholm operator of index zero.*

Proof. To show $r_I^* = r'_I$ one only needs $s(T \cdot H) \leqslant s(T)\, s(H)$ for T and H in $L(X)$ and $s(T) \geqslant 0$ for $T \in L(X)$. The rest is a standard argument. To show (ii), we use a lemma due to Nussbaum [10] which states that if $|\lambda| > r'_e$ then $\lambda - T$ is a Fredholm operator of index zero, where $r'_e = \lim_n (\gamma(T^n))^{1/n}$. By our assumption on the operator measure, we have $r'_I \geqslant r'_e$ which gives part (ii).

THEOREM I.8. *Let X be a Banach space and $T \in L(X)$. If the operator measure s satisfies the condition $s(T) \geqslant \gamma(T)$ or $s(T) \geqslant c(T)$, Then*

$$r_e(T) = \lim_n (s(T^n))^{1/n}$$

Proof. Let $r''_I = \lim_n \|[T^n]\|^{1/n}$ and $r_I = \sup\{|\lambda| : \lambda \in \sigma([T])\}$, where $[T]$ is the equivalence class of T in $L(X)/I$. From the spectral radius formula, we have $r_I = r''_I$.

Claim 1. $r'_I \leqslant r_I$, where $r'_I = \lim_n (s(T^n))^{1/n}$. Let $A \in I$. Then $s(T) = s(T - A + A) \leqslant s(T - A) + s(A) \leqslant \|T - A\|$ is true for every A and hence

$$s(T) \leqslant \inf_n \|T - A\| = \|T\|_I,$$

which gives Claim 1.

Claim 2. $r_I \leqslant r'_I$. Let $G = \{\lambda \in C : |\lambda| > r'_I\}$. Then G is an unbounded subset of the complex plane and $G \cap \delta(T) \neq \varnothing$. Let $\lambda \in G$, then by Lemma I.7, $\lambda - T$ is a Fredholm operator of index zero and hence $G \subset H_0$. Then by Proposition I.6, $\lambda \notin \sigma_b(T)$. Thus $G \cap \sigma_b(T) = \varnothing$; also $\sigma_I(T) \subset \sigma_e(T) \subset \sigma_b(T)$. Therefore $G \cap \sigma_I(T) = \varnothing$ and $r_I \leqslant r'_I$.

II. EXAMPLES OF OPERATOR MEASURES

Notation. Let $\mathscr{A}$ be the closed ideal with $\mathscr{F} \subset \mathscr{A} \subset \mathscr{K}$. For $T \in L(X)$, define

$$\|T\|_{\mathscr{A}} = d(T, \mathscr{A}) = \inf\{\|T - A\| : A \in \mathscr{A}\}.$$

PROPOSITION II.1. $\|T\|_{\mathscr{A}}$ *is the largest s-measure vanishing on* $\mathscr{A}$.

Proof. $\|T\|_{\mathscr{A}}$ is the norm of $[T]$ in $L(X)/\mathscr{A}$ and it is well known that this is a Banach algebra norm such that $\|T\|_{\mathscr{A}} \leqslant \|T\|$. Thus (1), (2), and (3) of Definition I.3 are immediate. Also $\|T\|_{\mathscr{A}} = 0$ implies $T \in \mathscr{A}$ and so $T \in \mathscr{K}$. Note that $\| \ \|_{\mathscr{A}}$ is the largest s-measure vanishing on $\mathscr{A}$. Indeed, suppose μ is another s-measure vanishing on $\mathscr{A}$ and $A \in \mathscr{A}$, then

$$\mu(T) = \mu(T - A + A) \leqslant \mu(A) + \|T - A\|$$

hence

$$\mu(T) \leqslant \|T - A\| \leqslant \|T\|_{\mathscr{A}} (1 + \varepsilon).$$

DEFINITION II.2. For $T \in L(X)$ and closed ideal $\mathscr{A}$ such that $\mathscr{F} \subset \mathscr{A} \subset \mathscr{K}$, let

$$\delta_{\mathscr{A}}(T) = \|TQ_X^1\|_{\mathscr{A}} \qquad \text{and} \qquad c_{\mathscr{A}}(T) = \|J_X^{\infty} T\|_{\mathscr{A}},$$

where $Q_X^1 : l_I^1 \to X$ denotes a surjection with $I = \mathscr{U}_X$, unit ball of X and $J_F^{\infty} : X \to l_I^{\infty}$ is an isometry with $I = \mathscr{U}_{X'}$, the ball of X', the dual space of X. Observe that $\delta_{\mathscr{A}}(T) = 0$ and $c_{\mathscr{A}}(T) = 0$ for $T \in \mathscr{A}$. An s-measure μ is called an injective s-measure if $\mu(T) = \mu(J_X^{\infty} T)$ and surjective s-measure if $\mu(T) = \mu(TQ_X^1)$.

PROPOSITION II.3. (i) *If $\delta_{\mathscr{A}}(T) = 0$, then T is compact.*

(ii) *If X has lifting property, then $\delta_{\mathscr{A}}(T) = \|T\|_{\mathscr{A}}$.*

(iii) *$\delta_{\mathscr{A}}(\cdot)$ is a surjective s-measure.*

(iv) *$\delta_{\mathscr{A}}(T) = 0$ if and only if $T \in (\bar{\mathscr{A}})^s$, where $(\bar{\mathscr{A}})^s$ denotes the surjective hull of the closure of the ideal $\mathscr{A}$. For definition of the surjective hull of an ideal see* [11].

Proof. (i) Since $\mathscr{A} \subset \mathscr{K}$ $\|TQ_X^1\|_{\mathscr{K}} \leqslant \|TQ_X^1\|_{\mathscr{A}}$ or, equivalently, $\delta_{\mathscr{K}}(T) \leqslant \delta_{\mathscr{A}}(T)$. However, from [2] we know $\delta_{\mathscr{K}}(T) = \gamma(T)$ and from [1] we have $\gamma(T) = \lim_n \delta_n(T)$ therefore if $\delta_{\mathscr{A}}(T) = 0$ then T is compact.

(ii) $\|\cdot\|_{\mathscr{A}}$ is the largest s-measure vanishing on $\mathscr{A}$, therefore $\delta_{\mathscr{A}}(T) \leqslant \|T\|_{\mathscr{A}}$. Let $B: X \to l_I^1$ be a map with $Q_X^1 B =$ identity on E and $\|B\| = 1$,

$$\delta_{\mathscr{A}}(T) \leqslant \|T\|_{\mathscr{A}} = \|TQ_X^1 B\|_{\mathscr{A}} \leqslant \|TQ_X^1\| \; \|B\| \leqslant \delta_{\mathscr{A}}(T).$$

(iii) l_I^1 has the lifting property so by (ii) we have

$$\|TQ_X^1\|_{\mathscr{A}} = \delta_{\mathscr{A}}(TQ_X^1).$$

(iv) Since by assumption $\mathscr{A}$ is closed, $\mathscr{A} = \bar{\mathscr{A}}$. Now let $T \in (\mathscr{A})^s$ then $TQ_X^1 \in \mathscr{A}$ and $\|TQ_X^1\|_{\mathscr{A}} = 0$. Conversely if $\delta_{\mathscr{A}}(T) = 0$ then $TQ_X^1 \in \mathscr{A}$ giving $T \in (\mathscr{A})^s$.

PROPOSITION II.4. (i) *If* $c_{\mathscr{A}}(T)=0$ *then* T *is compact.*

(ii) *If* X *has the extension property* $c_{\mathscr{A}}(T)=\|T\|_{\mathscr{A}}$.

(iii) $c_{\mathscr{A}}(\cdot)$ *is an injective s-measure.*

(iv) $c_{\mathscr{A}}(T)=0$ *if and only if* $T\in(\bar{\mathscr{A}})^i$, *where* $(\bar{\mathscr{A}})^i$ *denotes the injective hull of the closure of the ideal* $\mathscr{A}$. *For definition of the injective hull of an ideal see* [11].

Proof. (i) Since $\mathscr{A}\subset\mathscr{K}$ we have $\|J_F^\infty T\|_{\mathscr{K}}\leqslant\|J_F^\infty T\|_{\mathscr{A}}$ or equivalently $c_{\mathscr{K}}(T)\leqslant c_{\mathscr{A}}(T)$. But since $c_{\mathscr{K}}(T)=c(T)$ (see [8]) we see that T is compact.

(ii) $\|\cdot\|_{\mathscr{A}}$ being the largest s-measure vanishing on $\mathscr{A}$ gives $c_{\mathscr{A}}(T)\leqslant\|T\|_{\mathscr{A}}$. Let $B\colon l_I^\infty\to X$ be the map with $BJ_X^\infty=$ identity on X and $\|B\|\leqslant 1$,

$$\|T\|_{\mathscr{A}}=\|BJ_X^\infty T\|_{\mathscr{A}}\leqslant\|B\|\;\|J_X^\infty T\|_{\mathscr{A}}\leqslant c_{\mathscr{A}}(T).$$

(iii) l_I^∞ has the extension property so by (ii) we have

$$c_{\mathscr{A}}(J_X^\infty T)=\|J_X^\infty T\|_{\mathscr{A}}.$$

(iv) Let $T\in(\bar{\mathscr{A}})^i$ then $J_X^\infty T\in\mathscr{A}$ and hence $c_{\mathscr{A}}(T)=0$. The reverse implication follows from the definition.

REFERENCES

1. A. G. AKSOY, "Approximation schemes, related s-numbers and applications," Ph.D. thesis, University of Michigan, 1984.
2. K. ASTALA, On measures of non-compactness and ideal variations in Banach spaces, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, Dissertationes **29** (1980), 1–42.
3. F. V. ATKINSON, The normal solubility of linear equations in normed spaces, *Mat. Sb.* (*N.S.*) **28** (70) (1951), 3–14. [Russian]
4. F. E. BROWDER, On the spectral theory of elliptic differential operators, *Math. Ann.* **142**, (1961), 22–130.
5. A. M. DAVIE, The approximation problem for Banach spaces, *Bull. London Math. Soc.* **5** (1973), 261–266.
6. P. ENFLO, A counterexample to the approximation property in Banach spaces, *Acta Math.* **BO** (1973), 309–317.
7. I. C. GOHBERG AND M. G. KREIN, The basic propoitions on defect numbers, root numbers and indices of linear operators, *in* Amer. Math. Soc. Translations, Series 2 Vol. 13, pp. 185–264, Amer. Math. Soc., Providence, RI, 1960.
8. A. LEBOW AND M. SCHECHTER, Semigroups of operators and measures of noncompactness, *J. Funct. Anal.* **7** (1971), 1–26.
9. K. MATTILA, Normal operators and proper boundary points on the spectra of operators on a Banach space, *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes* **19** (1978).
10. R. O. NUSSBAUM, The radius of the essential spectrum, *Duke Math. J.* **37** (1970), 473–478.
11. A. PIETSCH, "Operator Ideals," North-Holland, Amsterdam, 1980.