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Q-COMPACT SETS AND Q-COMPACT MAPS

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Abstract. We shall introduce the notion of Q-compactness for an operator T between Banach spaces and consider the relationships between Q-compact sets and maps as well as measures of non-Q-compactness.

Introduction. The notion of an approximation scheme on a Banach space and its use in approximation theory can be found in Butzer and Scherer [2] and in Pietsch [6]. In the present paper we introduce a *refined notion of compactness* by developing a *refined notion of an approximation scheme* Q on a Banach space X. It is well-known that the Kolmogorov numbers $\delta_n(T)$ can be used to estimate the *degree of compactness* of an operator T between two Banach spaces. Generalized Kolmogorov numbers $\delta_n(T;Q)$ can be defined to obtain a further possibility for doing this. These numbers are a natural extension of the standard Kolmogorov numbers in the sense that

$$\delta_n(T;Q) = \delta_n(T)$$

in the case that Q is the set of all at-most-n-dimensional subspaces of X. A map $T \in L(X)$ is said to be Q*-compact* if $\lim_n \delta_n(T;Q) = 0$. First we provide an example of a Q-compact map which is not a compact, thus showing that Q-compactness is a genuine generalization of compactness. Then, taking the well-known characterization of compact sets as a model, a Dieudonne-Schwartz type representation theorem for Q-compact sets is obtained for a bounded set D in X. This representation leads to the concept of a measure $\gamma(D;Q)$ of non-Q-compactness, and it is proven that $\gamma(D;Q) = \lim_n \delta_n(D;Q)$. Furthermore, several properties of Q-compact maps and their relation to Q-compact sets are studied.

Preliminaries. I) Let X be a Banach space over the field K of real or complex numbers and N be the set of all non-negative integers. For each $n \in N$, let $Q_n = Q_n(X)$ be a family of subsets of X satisfying the following conditions:

(1) $\{0\} = Q_0 \subset Q_1 \subset \cdots \subset Q_n \subset \dots$;

(2) $\lambda Q_n \subset Q_n$ for every $n \in N$ and $\lambda \in K$;

(3) $Q_n + Q_m \subset Q_{n+m}$ for every $n, m \in N$.

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Then $Q(X) = (Q_n(X))_{n\in N}$ is called an *approximation scheme* on X. We shall simply use Q_n to denote $Q_n(X)$ if the context is clear.

Examples

1) Q_n = the set of all at-most-n-dimensional subspaces of any given Banach space X.

2) Let E be a Banach space and $X = L(E)$; let $Q_n = N_n(E)$, where $N_n(E)$ = the set of all n-nuclear maps [5] on E.

3) Let $\alpha^k = (\alpha_n)^{1+\frac{1}{k}}$, where (α_n) is a nuclear exponent sequence [3]. Then Q_n on $X = L(E)$ can be defined as the set of all $\Lambda_\infty(\alpha^k)$-nuclear maps on E.

II) Le U_X be the closed unit ball of X and D be a bounded subset of X. Then the n^{th} *generalized Kolmogorov number* $\delta_n(D;Q)$ of D with respect to U_X is defined by

$$\delta_n(D;Q) = \inf\{r > 0 : D \subset rU_X + A \text{ for some } A \in Q_n(X)\}.$$

The n^{th} Kolmogorov number $\delta_n(T;Q)$ of $T \in L(X)$ is defined as $\delta_n(T(U_X);Q)$.

From I) and II) it follows that $\delta_n(T;Q)$ forms a non-increasing sequence of non-negative numbers:

$$\| T \| = \delta_0(T;Q) \geq \delta_1(T;Q) \geq \cdots \geq \delta_n(T;Q) \geq 0.$$

III) A bounded subset D of X is said to be a *Q-compact set* if $\lim_n \delta_n(D;Q) = 0$ and $T \in L(X)$ is said to be a *Q-compact operator* if $\lim_n \delta_n(T;Q) = 0$, i.e. $T(U_X)$ is a Q-compact set.

1. Q-Compactness Does Not Imply Compactness. In this section we show that in $L_p[0,1], 2 \leq p \leq \infty$, with a suitably defined approximation scheme, we can find a Q-compact map which is not compact.

Let $[r_n]$ br the space spanned by the Rademacher functions. It can be seen from the Khinchin Inequality that [4]

$$l_2 \approx [r_n] \subset L_p[0,1] \text{for all } 1 \leq p < \infty.$$

We define an approximation scheme A_n on $L_p[0,1]$ as follows:

$$A_n = \{f \in L_p[0,1] : f \in L_{p+\frac{1}{n}}\} \text{ or simply } A_n = L_{p+\frac{1}{n}}.$$

$L_{p+\frac{1}{n}} \subset L_{p+\frac{1}{n+1}}$ gives us $A_n \subset A_{n+1}$ for $n = 1, 2, \ldots$, and it is easily seen that $A_n + A_m \subset A_{n+m}$ for $n, m = 1, 2, \ldots$, and that $\lambda A_n \subset A_n$. Thus $\{A_n\}$ is an approximation scheme in the sense of Pietsch [6].

Next we observe the existance of a projection

$$\underline{P} : L_p[0,1] \to R_p \text{ for } p \geq 2,$$

where R_p denotes the closure of the span of $\{r_n(t)\}$ in $L_p[0,1]$. We know that for $p \geq 2, L_p[0,1] \subset L_2[0,1]$. Now R_2 is a closed subspace of $L_2[0,1]$ and $\underline{P}_2 : L_2[0,1] \to R_2$ is an orthogonal projection onto R_2. Then $\underline{P} = j \circ \underline{P}_2 \circ i$, where i, j are isomorphisms shown in the diagram below, is clearly a projection.

$$\begin{array}{ccc} L_p & \xrightarrow{\ i\ } & L_2 \\ {\scriptstyle \underline{P}}\downarrow & & \downarrow{\scriptstyle \underline{P}_2} \\ R_p & \xleftarrow[\ j\ ]{} & R_2 \end{array}$$

Proposition 1. *For $p \geq 2$ the projection $\underline{P} : L_p[0,1] \to R_p$ is Q-compact but not compact.*

Proof. Let U_{R_p}, U_{L_p} denote the closed unit balls of R_p and L_p, respectively. It is easily seen that $\underline{P}(U_{L_p}) \subset \| \underline{P} \| U_{R_p}$. But $U_{R_p} \subset CU_{R_{p+\frac{1}{n}}}$ where C is a constant follows from the Khinchin inequality. Therefor $\underline{P}(U_{L_p}) \subset L_{p+\frac{1}{n}}$, which gives $\delta_n(P,Q) \to 0$. To see that $\underline{P}$ is not a compact operator, observe that $\dim R_p = \infty$ and $I - \underline{P}$ is projection with kernel R_p, so $I - \underline{P}$ is not a Fredholm operator. Therefore $\underline{P}$ is not a Riesz operator, but every compact operator is a Riesz operator (see [5]) so $\underline{P}$ cannot be a compact operator.

2. Q-Compactness of Bounded Sets in a Banach Space. Let X be a Banach space. A bounded subset D of X is said to be *Q-compact* if $\delta_n(D;Q) \to 0$ $(n \to \infty)$. We assume each $A_n \in Q_n (n \in N)$ is separable. It is immediate from the definitions that Q-compact sets are separable and Q-compact maps have separable range. A sequence $(x_{n,k})_k \subset A_n$ is said to be an order-c_0-sequence if the following hold:

(1) for every $n \in N$ there exists an $A_n \in Q_n$ and $(x_{n,k})_k \subset A_n$;

(2) $\| x_{n,k} \| \to 0$ as $n \to \infty$ uniformly in k.

Theorem 2. *Suppose (X, Q_n) is an approximation scheme with sets $A_n \in Q_n$ assumed to be solid (i.e., $|\lambda| A_n \subset A_n$ for $|\lambda| \leq 1$). Then a bounded subset D of X is Q-compact if and only if there exists an order-c_0-sequence $(x_{n,k})_k \subset A_n$ such that*

$$D \subset \{\sum_{n=1}^{\infty} \lambda_n x_{n,k(n)} : x_{n,k(n)} \in (x_{n,k}) \sum_{n=1}^{\infty} |\lambda_n| \leq 1\}.$$

Proof. Let D be Q-compact. Then $\delta_n(2D, Q) \to 0$ and so there exists n_1 such that

$$2D \subset \frac{1}{4}U + A_{n_1}.$$

Since A_{n_1} is separable let $(x_{1,k})$ be a countable dense subset of A_{n_1}; then it is easy to see that $B_1 = (2D + \frac{1}{2}U) \cap ((x_{1,k})_k) \neq \phi$ (and is countable) and $2D \subset B_1 + \frac{1}{2}U$.

Let $D_1 = (2D - B_1) \cap \frac{1}{2}U$, where $2D - B_1$ is the ordinary vector difference. Then D_1 is a bounded set (being in $\frac{1}{2}U$) and given $\epsilon > 0$ we get, by the Q-compactness of $2D$, that $2D - B_1 \subset \epsilon U + A_m + \tilde{A}_{n_1} \subset \tilde{\tilde{A}}_{m+n_1} + \epsilon U$ for a suitable m and suitable $\tilde{A}_{n_1} \in Q_{n_1}$; this is true because $B_1 \subset \tilde{A}_{n_1}$ and $\lambda \tilde{A}_{n_1} \in Q_{n_1}$ for each λ. This shows that D_1 is Q-compact, and as before there exists A_{n_2} such that $2D_1 \subset \frac{1}{8}U + A_{n_2}$; let $(x_{2,k})$ be a dense subset of A_{n_2} Then.

$$B_2 = (2D_1 + \frac{1}{4}U) \cap ((x_{2,k})_k) \text{ is non-empty;}$$

$$2D_1 \subset B_2 + \frac{1}{4}U;$$

$$D_2 = (2D_1 - B_2) \cap \frac{1}{4}U \text{ is } Q\text{-compact.}$$

Continuing this process we define

$$B_m = (2D_{m-1} + \frac{1}{2^m}U) \cap ((x_{m,k})_k), (x_{m,k}) \text{ dense in } A_{n_m};$$

then $2D_{m-1} \subset B_m + \frac{1}{2^m}U$ and we define

$$D_m = (2D_{m-1} - B_m) \cap \frac{1}{2^m}U.$$

Our construction gives for each $d \in D$, successively chosen $b_i \in B_i, i = 1, 2, \ldots, k$ such that

$$d - (\frac{1}{2}b_1 + \frac{1}{2^2}b_2 + \cdots + \frac{1}{2^k}b_k) \in 2^{-k}D_k,$$

and since $D_k \subset 2^{-k}U$, it follows that

$$d = \sum_{n=1}^{\infty} \frac{1}{2^n} b_n.$$

Since each $b_n = x_{n,k(b)}$ for a suitable $k(b)$ and since $b_n \in B_n \subset 2D_{n-1} + \frac{1}{2^n}U \subset 2 \cdot \frac{1}{2^{n-1}}U + \frac{1}{2^n}U \subset \frac{3}{2^{n-2}}U$ it follows that $\| b_n \| \to 0$.

In the reverse direction, suppose we have for each n an $A_n \in Q_n$ and $(x_{n,k})_k \subset A_n$ with $\| x_{n,k} \| \to 0$ as $n \to \infty$ uniformly in k and

$$D \subset \{\sum_n \lambda_n x_{n,k(n)} : \sum_n |\lambda_n| \leq 1\} = C, \text{ say.}$$

Since for each $c \in C$ we can write

$$c = \sum_{n=1}^{m} \lambda_n x_{n,k(n)} + \sum_{n=m+1}^{\infty} \lambda_n x_{n,k(n)} = u + v,$$

where $u \in \lambda_1 A_1 + \cdots + \lambda_m A_m$, our assumptions on (Q_n) and the solidness of the A_n 's give that $u \in \tilde{A}_{m^2}$; also, given $\epsilon > 0$ we may choose m such that $\| x_{n,k} \| < \epsilon$ for each $k > m$. Thus $C \subset \epsilon U + \tilde{A}_{m^2}$ and so $\delta_n(C, Q) \to 0$ as $n \to \infty$, and therefor also $\delta_n(D, Q) \to 0$.

Remarks. i) Thgeorem 2 can be considered as an analogue of the Dieudonne-Schwartz lemma on compact sets in terms of standard Kolmogorov diameter. If one chooses Q_n to be the at-most-n-dimensional subspaces of X one can show that Q-compactness of a bounded subset D coincides with the usual definition of compactness of D.

ii) The author and M. Nakamura have proven a similar theorem for p-normed spaces, $0 < p \leq 1$ [1].

3. Q-compact Maps. For a given approximation scheme Q_n on X we shall define a continuous linear map $T \in L(X)$ to be Q-compact if $T(U_X)$ is Q-compact in X or equivalently if $\lim_n \delta_n(T(U_X); Q) = \lim_n \delta_n(T; Q) = 0$.

Let $\mathcal{A}$ be the ideal defined as

$$\mathcal{A} = \{T \in L(X) : \delta_n(T; Q) \to 0 \text{ as } n \to \infty\}$$

and let $\mathcal{A}^s$ denote the surjective hull of $\mathcal{A}$, which is defined by

$$\mathcal{A}^s = \{T \in L(X) : \delta_n(TQ_{E^1}; Q) \to 0 \text{ as } n \to \infty\}.$$

where Q_{E^1} is a surjection of l_I^1 onto X with $Q_{E^1}(U_{l_I^1}) = U_X$.

Proposition 3.

i) *Q-compact maps have separable range;*

ii) *the uniform limit of Q-compact maps is Q-compact;*

iii) *an ideal of Q-compact maps is equal to its surjective hull, i.e. $\mathcal{A} = \mathcal{A}^s$.*

Proof. i) follows from the definition. For ii) we first observe that $\delta_0(T; Q) \leq ||T||$. Now suppose (T_n) is a sequence of Q-compact maps, and let $T = \lim_n T_n$.

Then

$$\delta_n(T; Q) = \delta_n(T - T_n + T_n; Q) \leq \delta_0(T - T_n; Q) + \delta_n(T_n; Q)$$
$$\leq ||T - T_n|| + \delta_n(T_n; Q)$$

gives that T is Q-compact too.

For iii), $\mathcal{A} \subset \mathcal{A}^s$ follows from the fact that

$$\delta_n(TQ_{E^1}; Q) \leq \delta_n(T; Q)||Q_{E^1}|| = \delta_n(T; Q);$$

on the other hand

$$\delta_n(TQ_{E^1}; Q) = \delta_n(TQ_{E^1}(U_{l_I^1}); Q) = \delta_n(T; Q);$$

gives the equality readily.

Next we give a characterization of Q-compact subsets of X via Q-compact maps into X.

Theorem 4. Assume (X, Q_n) is an approximation scheme on the Banach space X with each $A_n \in Q_n$ being a vector subspace of X. Then a bounded subset D of X is Q-compact if and only if $D \subset T(U_E)$ for a suitable Banach space E and a Q-compact map T on E into X.

Proof. We need only prove the "only if" part. Let D be Q-compact and let C denote the closed, absolute convex hull of D. Then that C is Q-compact is easily seen as follows: each $c \in C$ is of the form $c = \sum_{i=1}^{m} \lambda_i d_i$, with $\sum_{i=1}^{m} |\lambda_i| \leq 1$ and $d_i \in D$ for each i; given $\epsilon > 0$, there exists N such that for all $n \geq N, \delta_n(D, Q) < \epsilon$ and equivalently $D \subset \epsilon U_X + A_N$ and obviously then $C \subset \epsilon U_X + A_n$.

Let X_C denote the linear subspace of X spanned by the elements of C endowed with the norm given by the gauge (=Minkowski functional) μ of C. Then (X_C, μ_C) is a Banach space. Let $E = (X_C, \mu_C)$. If T is the canonical injection of X_C into X, then $T(U_E) = C \supset D$ and T is Q-compact.

4. Measures of Non-Q-Compactness. Let X be a Banach space and D be a bounded subset of X. Assume that each $A_n \in Q_n$ $(n \in N)$ is solid. The ball measure of noncompactness of D, denoted by $\gamma(D)$, is defined by

$$\gamma(D) = \inf\{r > 0 : D \subset \bigcup_{i=1}^{k} B(x_i, r)\},$$

where $B(x_i, r)$ stands for the ball centered at $x_i \in X$ with radius r and k is arbitrary but finite.

Suppose $(x_{n,k})_k$ is an order-c_0-sequence in X as defined in section 2. Then S_m, associated with $(x_{n,k})_k$, is defined by

$$S_m = \{\sum_{n=1}^{m} \lambda_n x_{n,k(n)} : \sum_{n=1}^{m} |\lambda_n| \leq 1\}$$

where $x_{1,k(1)} \in A_1, x_{2,k(2)} \in A_2, \ldots, x_{m,k(m)} \in A_m$. Then $S_m \subset A_1 + A_2 + \cdots + A_m \in Q_{m^2}$. So if Q_n is n-dimensional, S_n is at most n^2-dimensional.

For a bounded set D in X, we define the *ball measure of non-Q-compactness* $\gamma(D, Q)$ of D by

$$\gamma(D, Q) = \inf\{r > 0 : \exists \text{ order-}c_0\text{-sequence } (x_{n,k})_k \text{ and assosiated } S_n$$

$$\text{such taht } D \subset \bigcup_{x \in S_n} B(x, r) \text{for some } n\}.$$

The following proposition defines the ball measure of non-Q-compactness as a limit of the Kolmogorov diameter of D defined with respect to the given approximation scheme.

Theorem 5. *Let X be a Banach space with approximation scheme Q_n and let D be a bounded subset of X; then*

$$\gamma(D, Q) = \lim_{n \to \infty} \delta_n(D; Q).$$

Proof. Let r be admissible for $\gamma(D,Q)$, then there exists an order-c_0-sequence $(x_{n,k})$ and associated (S_n) such that

$$D \subset \bigcup_{x\in S_n} B(x,r) = \bigcup_{x\in S_n} \{x + rU_X\}.$$

Now $S_n \subset \tilde{A}_{n^2} \in Q_{n^2}$ and if $m \geq n^2$ we have $S_n \subset \tilde{A}_m \in Q_m$; therefore r is admissible for $\delta_m(D,Q)$ and hence $\gamma(D,Q) \geq \delta_m(D,Q)$.

Suppose $\inf \delta_n(D,Q) = \mu < \lambda$. Then there exists n such that $\delta_n(D,Q) < \lambda$ so there exists $\lambda' < \lambda$ and A_n such that

$$D \subset \lambda' U + A_n.$$

Let $D \subset K + L$, where $K \subset \lambda' U$ and $L \subset A_n$. Since $L \subset A_n$ and $\delta_i(L,Q) \to 0$, hence by Theorem 2 there exists an order-c_0-sequence $(x_{n,k})_k$ such that

$$L \subset \{\sum_{n=1}^{\infty} \lambda_n x_{n,k(n)} : \sum_{n=1}^{\infty} |\lambda_n| \leq 1\}. \qquad (*)$$

Because $(x_{n,k})_k$ is an order-c_0-sequence, given $\epsilon > 0$ we can find N such that $||x_{n,k}|| \leq \epsilon$ for all $n \geq N$ and all k. Using equation $(*)$ above, we can write every $l \in L$ as

$$l = \sum_{1}^{N} \lambda_n x_{n,k(n)} + \sum_{N+1}^{\infty} \lambda_n x_{n,k(n)}.$$

It is easily follows that $l = x + \epsilon U_X$ for some $x \in S_N$. Hence $||l - x|| < \epsilon$ and $L \subset \bigcup_{x\in S_N} B(x,\epsilon)$. Therfore $D \subset \lambda' U + \bigcup_{x\in S_N} B(x,\epsilon) \subset \bigcup_{x\in S_N} B(x,\lambda'+\epsilon) \subset \bigcup_{x\in S_N} B(x,\lambda+\epsilon)$. Hence $\gamma(D,Q) \leq \epsilon + \lambda$ and $\gamma(D,Q) \leq \lim_n \delta_n(D;Q)$.

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