Modular spaces and K-widths

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Abstract. In this paper, we show that the ball measure of noncompactness of a modular space X_ρ is equal to the limit of its K-widths when ρ is a left continuous, s-convex modular function, without any Δ_2-condition. We also obtain a similar result for SF-spaces, when the SF-norm $\mathcal{N}$ is uniformly continuous.

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1. Notation and definitions

Throughout the following X is a linear space over a field K ($K = \mathbb{R}$ or $K = \mathbb{C}$).

I. A function $\rho : X \to [0, \infty]$ is called *modular* if the following hold for arbitrary $x, y \in X$:

1. $\rho(x) = 0$ iff $x = 0$.
2. $\rho(\alpha x) = \rho(x)$ if $\alpha \in K$, $|\alpha| = 1$.
3. $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for $\alpha,\ \beta \geq 0, \alpha + \beta = 1$.

If in place of 3) we have

$$\rho(\alpha x + \beta y) \leq \alpha^s \rho(x) + \beta^s \rho(y) \quad \text{for } \alpha,\ \beta \geq 0,\ \alpha^s + \beta^s = 1$$

for an $s \in (0, 1]$, then ρ is called s-convex modular (convex if $s = 1$).

The set

$$X_\rho = \{x \in X : \lim_{\alpha \to 0} \rho(\alpha x) = 0\}$$

is called a modular space.

Each modular space X_ρ may be equipped with an F-norm given by the formula

$$|x|_\rho = \inf\{u > 0 : \ \rho(x/u) \leq u\} \quad \text{for } x \in X_\rho.$$

A modular ρ in X is called left continuous if $\lim_{\lambda \to 1^-} \rho(\lambda x) = \rho(x)$ for all $x \in X_\rho$.

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It is known (see [8]) that, if ρ is left continuous, s-convex modular in X, then the inequalities $|x|_\rho^s \leq 1$ and $\rho(x) \leq 1$ are equivalent for every $x \in X_\rho$. Here by $|x|_\rho^s$ we mean

$$|x|_\rho^s = \inf\{u > 0 : \rho(\frac{x}{u^{1/s}}) \leq 1\}.$$

A particular class of modular spaces are Orlicz–Musielak spaces. To define these spaces, let (E, Σ, μ) be a measure space and let $f : E \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the following conditions.

1. $f(t, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing, continuous function such that $f(t, 0) = 0$ and $f(t, u) > 0$ for $u > 0$.
2. $f(\cdot, u) : E \to \mathbb{R}_+$ is a Σ-measurable function for all $u > 0$.
3. $\int_A f(t, u)\, d\mu(t) < \infty$ for every $u > 0$ and $A \in \Sigma$, $\mu(A) < \infty$.

Suppose that X is the space of all real (or complex) valued Σ-measurable functions defined on E. For $x \in X$, set

$$\rho_f(x) = \int_E f(t, |x(t)|)\, d\mu(t).$$

From the above 1) and 2), it is clear that ρ_f is modular. The modular space X_ρ is called Orlicz–Musielak space (and Orlicz space, if the function f is independent of the variable t).

We say the function f satisfies a Δ_2-condition if

$$f(t, 2u) \leq K f(t, u) + h(t)$$

for all $u \geq 0$, $t \in E$ where $h \in L^1(E, \mu)$, $h \geq 0$ and K is a positive constant independent of the variables t, u.

For further theory of modular spaces we refer to [8].

II. Assume a function $\mathcal{N} : X \to \mathbb{R}_+$ satisfies the following conditions.

1. $\mathcal{N}(x) = 0$ iff $x = 0$.
2. $a_n \to a$ and $\mathcal{N}(x_n - x) \to 0$ then $\mathcal{N}(a_n x_n - ax) \to 0$ for all sequences $\{a_n\} \subset K$ and $\{x_n\} \subset X$.
3. If $\mathcal{N}(x_n - x) \to 0$ and $\mathcal{N}(y_n - y) \to 0$ then $\mathcal{N}(x_n + y_n - x - y) \to 0$ for all sequences $\{x_n\}$, $\{y_n\} \subset X$.
4. $\mathcal{N}(ax) = \mathcal{N}(x)$ for every $x \in X$ and $a \in K$, $|a| = 1$;
5. $\mathcal{N}(x_n - x) \to 0$ then $\mathcal{N}(x_n) \to \mathcal{N}(x)$ for every $\{x_n\} \subset X$.
6. The space X is complete with respect to the topology induced by the family $C = \{B(x; r) : x \in X,\ r > 0\}$ where

$$B(x; r) = \{y \in X : \mathcal{N}(x - y) < r\}.$$

The pair $(X, \mathcal{N})$ is said to be an SF-space and the function $\mathcal{N}$ will be called an SF-norm.

Note that each F-space (in particular each Banach space) is an SF-space. If f satisfies a Δ_2-condition, then each Orlicz–Musielak space (X_{ρ_f}, ρ_f) is an SF-space. However, there exist SF-spaces which are neither F-spaces nor modular spaces as can be seen by the following

Example ([7]). Assume $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following conditions:

1. f is continuous and $f(t) = 0$ iff $t = 0$.
2. There exists $d > 0$ such that $f_{|[0,d]}$ is strictly increasing.
3. There exists $d_1 > 0$ and $M > 1$ such that
$$f(t+s) \le M(f(t) + f(s)) \text{ for } s,\ t \in [0, d_1].$$
4. There exists $\lim_{t\to\infty} f(t) \in (0, \infty)$.

Let (E, Σ, μ) be a measure space, $\mu(E) < \infty$ and let $M(E) = \{x : E \to R,$ x is Σ-measurable $\}$. For $x \in M(E)$ define $\mathcal{N}(x) = \int_E f(|x(t)|)\, d\mu(t)$. By Fatou's lemma, the Riesz and the Lebesgue theorems, one can show that the pair $(M(E), \mathcal{N})$ is an SF-space. It is clear that for nonmonotonic f the function $\mathcal{N}$ is not modular, and if we additionally assume

$$\lim_{t\to+\infty} f(t) < \frac{1}{2} \sup\{f(t) : t \in \mathbb{R}_+\}$$

then the pair $(M(E), \mathcal{N})$ can not be an F-space.

The SF-norm $\mathcal{N}$ is called *nondecreasing* iff for any $t_1,\ t_2 \in \mathbb{R}$, $|t_1| \le |t_2|$ implies $\mathcal{N}(t_1 x) \le N(t_2 x)$ and given D in an SF-space $(X, \mathcal{N})$, we say D is bounded iff $\lambda_n \to 0$, $x_n \in D$ implies $\mathcal{N}(\lambda_n x_n) \to 0$. Analogously $D \subset X_\rho$ is *ρ-bounded* iff $a_n \to 0$, $x_n \in D$ implies $\rho(a_n x_n) \to 0$. The SF-norm $\mathcal{N}$ is called *uniformly continuous* if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\text{if } N(f-g) < \delta \text{ then } |N(f) - N(g)| < \varepsilon.$$

Note that in the definition of the SF-space, $\mathcal{N}$ is only a continuous function.

Proposition 1. *Let $(X, \mathcal{N})$ be an SF-space with $\mathcal{N}$ being uniformly continuous. Then for every $\varepsilon > 0$ there is V, an open neighbourhood of 0, such that*

$$B^{\mathcal{N}}(0, r) + V \subset B^{\mathcal{N}}(0, r + \varepsilon) \qquad (*)$$

where $B^{\mathcal{N}}(x, r) = \{y \in X : \mathcal{N}(x - y) \le r\}$.

The proof is straightforward.

III. Let $(X, \mathcal{N})$ be an SF-space. For $V \subset X$, $V \setminus \{0\} \neq \emptyset$ put

$$R_{\mathcal{N}}(V) = \inf\{\sup\{\mathcal{N}(tv) : t \in \mathbb{R}_+\} : v \in V \setminus \{0\}\}.$$

This number which may be equal to $+\infty$, is called the radius of the set V ([7]). As shown in the following example, it may also occur that $R_{\mathcal{N}}(V) = 0$.

Example. Let X be the space of all complex sequences equipped with an SF-norm $\mathcal{N}$ defined by

$$\mathcal{N}(x) = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i \frac{|x_i|}{1 + |x_i|} \quad \text{for } x \in X.$$

Let for $n \in N$, $V_n = \{x \in X : x_j = 0 \text{ for } j > n\}$. Then clearly $R_{\mathcal{N}}(V_n) = 2^{-n}$ and consequently $R_{\mathcal{N}}(X) = 0$.

In [7] it is shown that if X_ρ is a modular space with ρ s-convex modular, then $R_\rho(X_\rho) = +\infty$.

K-widths are extensively studied in the context of approximation theory [10]. Our aim in this paper is to connect K-widths with measures of noncompactness. Such connections are not only useful in fixed point theory (see [3], [11]) but also in the study of the radius of the essential spectrum (see [6], [9]). Measures of noncompactness for Orlicz spaces are studied in [1], [2] and [4]. In [5] one can find fixed point theorems for Orlicz modular spaces.

2. K-widths in SF-spaces

Let $(X, \mathcal{N})$ be an SF-space, D be bounded set in X. Then the ball measure of noncompactness of D, $\alpha(D)$, is

$$\alpha(D) = \inf\{r > 0 : D \subset \bigcup_{k=1}^{n} B^{\mathcal{N}}(x_k, r)\}$$

and K-th width of D, $d^k(D)$, is defined as

$$d^k(D) = \inf\{r > 0 : D \subset B^{\mathcal{N}}(0, r) + A_k \ \dim(A_k) \leq k\}.$$

Theorem 1. *Let D be a bounded subset of an SF-space $(X, \mathcal{N})$. Suppose that $\mathcal{N}$ is nondecreasing, $R_{\mathcal{N}}(X) = \infty$ and for every $r > 0$, $\varepsilon > 0$ and E finite dimensional subspace of X, there is an open neighborhood V of 0 in E such that*

$$B^{\mathcal{N}}(0, r) + V \subset B^{\mathcal{N}}(0, r + \varepsilon). \qquad (*)$$

Then $\alpha(D) = \lim_{k \to \infty} d^k(D)$.

Proof. Let D be a fixed bounded set in X. If there is $r > 0$ and $k \in N$ such that $D \subset \bigcup_{i=1}^{k} B^{\mathcal{N}}(x_i, r)$, then

$$D \subset B^{\mathcal{N}}(0,r) + \bigcup_{i=1}^{k}\{x_i\} \subset B^{\mathcal{N}}(0,r) + \mathrm{Span}(x_1, x_2, \ldots, x_k).$$

Therefore, if $\alpha(D) < \infty$ then $d = \lim_k d^k(D)$ is finite and $d \leq \alpha(D)$.

To obtain the other inequality, assume that $d < +\infty$, fix $k \in N$, $\varepsilon > 0$ and $A_k \subset X$ with $\dim A_k = k$ such that

$$D \subset B^{\mathcal{N}}(0, d^k(D) + \varepsilon) + A_k.$$

Let us define

$$D_1 = \{g \in B^{\mathcal{N}}(0, d^k(D) + \varepsilon) : \text{ there is } h \in A_k,\ g + h \in D\}$$
$$D_2 = \{h \in A_k : \text{ there is } g \in B^{\mathcal{N}}(0, d^k(D) + \varepsilon),\ g + h \in D\}.$$

Obviously $D \subset D_1 + D_2$. Now we claim that D_2 is a bounded set in $(X, \mathcal{N})$. Assume on the contrary that there is $\{h_n\} \subset D_2$ and $\{\lambda_n\} \subset \mathbb{R}$, $\lambda_n \to 0$ such that $\mathcal{N}(\lambda_n h_n)$ does not tend to zero. Since $\dim(A_k) = k$, one has

$$\lambda_n h_n = \sum_{i=1}^{k} r_i^n y_i$$

where $y_1, \ldots, y_k$ is a fixed basis of A_k. Consider the following cases.

Case 1: $\sup_{n\in N}(\max_{1\leq i\leq k} |r_i^n|) < +\infty$. Then passing to a subsequence, if necessary, by the properties of $\mathcal{N}$ (II.2 and 3) we may assume $\mathcal{N}(\lambda_n h_n - h) \to 0$ for some $h \in X$, $h \neq 0$.

On the other hand, by the definition of D_2, for every $n \in N$ there is $g_n \in D_1$ with $f_n = g_n + h_n \in D$. Since $R_{\mathcal{N}}(X) = +\infty$, there is a $t > 0$ such that $\mathcal{N}(th) > d^k(D) + \varepsilon$. By the boundedness of D, $\mathcal{N}(\lambda_n f_n) \to 0$. By II.1, II.2 and II.5 of the definition,

$$\mathcal{N}(t\lambda_n g_n) \to \mathcal{N}(th) > d^k(D) + \varepsilon.$$

But for $n \geq n_0$, $t\lambda_n < 1$. Since $\mathcal{N}$ is nondecreasing

$$\mathcal{N}(t\lambda_n g_n) \leq \mathcal{N}(g_n) \leq d^k(D) + \varepsilon \quad \text{for } n \geq n_0,$$

a contradiction.

Case 2: there is $i_0 \in 1, \ldots, k$ with $\lim_{n\to\infty} \inf |r_{i_0}^n| = +\infty$. Passing to a subsequence, if necessary, we may assume that for each $i \in \{1, 2, \ldots, k\}$,

$$c_i = \lim_n \frac{r_i^n}{r_{i_0}^n} \qquad (c_{i_0} = 1).$$

By the Definition II.1, we have

$$\mathcal{N}\left[\left(\frac{\lambda_n}{r_{i_0}^n}\right) h_n - \sum_{i=1}^{k} c_i y_i\right] \to 0.$$

We can set $h = \sum_{i=1}^{k} c_i y_i$, since $y_1, \ldots, y_k$ is a basis of A_k with $c_{i_0} = 1$, $h \neq 0$. Considering the subsequence $\lambda_n / r_{i_0}^n$ instead of λ_n and reasoning as in Case 1, we obtain a contradiction.

Therefore D_2 is a bounded set with respect to the topology defined by $\mathcal{N}$. Since each SF-space, as a complete, topological linear space with a countable basis of neighborhoods of 0 is metrizable, the SF-space $(A_k, \mathcal{N})$ is metrizable. But A_k is finite dimensional, the topology induced by $\mathcal{N}$ in A_k is the same as any norm topology in A_k. Hence, D_2 is bounded in any norm in A_k. Consequently $\bar{D}_2$ (the closure of D_2 in A_k in any norm) is a compact set.

Now by the assumption $(*)$, there is V an open neighborhood of 0 in A_k such that

$$B^{\mathcal{N}}(0, d^k(D) + \varepsilon) + V \subset B^{\mathcal{N}}(0, d^k(D) + 2\varepsilon).$$

Since $\bar{D}_2$ is a compact set $\bar{D}_2 \subset \bigcup_{i=1}^{l} x_i + V$. Note that

$$\begin{aligned} D &\subset D_1 + D_2 \\ &\subset B^{\mathcal{N}}(0, d^k(D) + \varepsilon) + \bigcup_{i=1}^{l} \{x_i + V\} \\ &\subset \bigcup_{i=1}^{l} \{x_i + B^{\mathcal{N}}(0, d^k(D) + 2\varepsilon)\} \\ &= \bigcup_{i=1}^{l} B^{\mathcal{N}}(x_i, d^k(D) + 2\varepsilon). \end{aligned}$$

Consequently, $\alpha(D) \le d^k(D) + 2\varepsilon$ for every $\varepsilon > 0$ and $k \in N$, which yields

$$\alpha(D) \le d = \lim_k d^k(D).$$

Remark. Condition (*) in Theorem 1 is satisfied if $\mathcal{N}$ is uniformly continuous, which covers the case of F-spaces. Moreover, if one assumes $\lim_{u \to \infty} f(t, u) = +\infty$ then the corresponding ρ_f F-norm $|\cdot|_\rho$ satisfies the assumptions in Theorem 1.

3. *K*-widths in modular spaces

Let X_ρ by a modular space and $D \subset X_\rho$ be a ρ-bounded set. Let B_ρ be the closed ρ-ball in X_ρ i.e., $B_\rho = \{x \in X_\rho : \rho(x) \le 1\}$. K-width of D in X_ρ, $d_\rho^k(D)$, defined as

$$d_\rho^k(D) = \inf\{\lambda > 0 : D \subset \lambda B_\rho + H : \dim H \le k\},$$

and ρ-ball measure of noncompactness of D, $\alpha_\rho(D)$, defined as

$$\alpha_\rho(D) = \inf\{\lambda > 0 : D \subset \bigcup_{i=1}^{k} \lambda(x_i + B_\rho) : x_1, \dots, x_k \in X_\rho\}.$$

Note that if ρ is a norm, then the notations d_ρ^k and α_ρ above coincide with the classical definitions of d^k and α.

The following are some basic properties of $\alpha_\rho(D)$.

Proposition 2. a) *Montonicity: $D_1 \subset D_2$ implies $\alpha_\rho(D_1) \le \alpha_\rho(D_2)$.*
b) *Semi-additivity: $\alpha_\rho(D_1 \bigcup D_2) = \max\{\alpha_\rho(D_1), \alpha_\rho(D_2)\}$.*
c) *Invariance under translation: $\alpha_\rho(D + x_0) = \alpha_\rho(D)$ for any $x_0 \in X_\rho$.*
d) *Algebraic semi-additivity: If ρ is convex, then $\alpha(D_1 + D_2) \le \alpha(D_1) + \alpha_\rho(D_2)$.*
e) *$\alpha_\rho(D) = \alpha_\rho(\overline{D})$ where the closureclosure is taken with respect to F-norm. Moreover if ρ is convex, then $\alpha(D) = \alpha(c_0(D))$ where $c_0(D)$ denotes the convex hull of D.*
f) *If B_ρ is a $|\cdot|_\rho$-bounded set, then $\alpha_\rho(D) = 0$ iff D is $|\cdot|_\rho$- compact.*

Proof. a) through e) follow from the definition. To prove f), set $B_{|\cdot|_\rho}(0, r) = \{x \in X_\rho : |x|_\rho \le r\}$ and $\alpha_{|\cdot|_\rho}(D) = \inf\{r > 0 : D \subset \bigcup_{i=1}^{k} [x_i + B_{|\cdot|_\rho}(0, r)]\}$, we claim that $\alpha_\rho(D) = 0$ implies $\alpha_{|\cdot|_\rho}(D) = 0$ and hence D is $|\cdot|_\rho$-compact. Since B_ρ is bounded set, $\lambda_0 \cdot B_\rho \subset B_\rho(0, r)$ for some λ_0. Then since $D \subset \bigcup_{i=1}^{k}\{x_i + \lambda_0 \cdot B_\rho\} \subset \bigcup_{i=1}^{k}\{x_i + B_\rho(C, r)\}$, we have the desired result. On the other hand, if D is $|\cdot|_\rho$-compact set, then for every $\lambda > 0$, $D \subset \bigcup_{i=1}^{k}\{x_i + \lambda \cdot \mathrm{int}(B_\rho)\}$. Hence, for every $x \in D$ contained in the open set $x_i + \lambda \cdot \mathrm{int}(B_\rho)$ since λ was arbitrary $\alpha_\rho(D) = 0$.

Theorem 2. *Suppose ρ is a modular and $R_\rho(X_\rho) = +\infty$. Suppose that the Luxemburg norm $|\cdot|_\rho$ satisfies the following conditions:*

1. *For every $\varepsilon > 0$, there is $\delta > 0$ with*
$$B_{|\cdot|_\rho}(0, 1 + \delta) \subset (1 + \varepsilon)\dot{B}_{|\cdot|_\rho}(0, 1).$$
2. $|x_n| \to 0$ *iff* $\rho(x_n) \to 0$.
3. $\lim_{c \to 1} \rho(cx) = \rho(x)$ *for* $x \in X_\rho$.

Then $\lim_{k \to \infty} d_\rho^k(D) = \alpha(D)$.

Proof. As in the proof of Theorem 1, one can easily establish that $\lim_k d_\rho^k(D) \le \alpha(D)$.

To prove the converse, consider $\varepsilon > 0$ and λ such that

$$\lim_k d_\rho^k(D) < \lambda < \lim_k d_\rho^k(D) + \varepsilon.$$

Define analogously as in Theorem 1,

$$\begin{aligned} D_1 &= \{x \in B_{|\cdot|_\rho}(0,1) : \quad \text{there is} \quad h \in H_{k_0} \quad \text{with} \quad \lambda x + h \in D\}; \\ D_2 &= \{h \in H_{k_0} : \quad \text{there is} \quad x \in B_\rho, \lambda x + h \in D\}. \end{aligned}$$

Note that by the Condition 3) in the Theorem 2 above, $B_\rho = B_{|\cdot|_\rho}(0,1)$. Let H_{k_0} be a subspace of X_ρ with $\dim H_{k_0} \le k_0$ and $D \subset \lambda B_\rho + H_{k_0}$. Reasoning as in Theorem 1, together with the Conditions 2) and 3) above, one can show that D_2 is a bounded set.

Now for $\varepsilon > 0$, choose $\delta > 0$ such that Condition 1) in the above Theorem 2 is satisfied. Since D_2 is a bounded set, $D_2 \subset \bigcup_{i=1}^k \lambda B_{|\cdot|_\rho}(x_i, \delta)$. Therefore,

$$\begin{aligned} D &\subset \lambda D_1 + D_2 \\ &\subset \lambda B_\rho + \bigcup_{i=1}^k \lambda B_{|\cdot|_\rho}(x_i, \delta) \\ &= \lambda[B_\rho + \bigcup_{i=1}^k \{B_{|\cdot|_\rho}(0,\delta) + x_i\}] \\ &\subset \lambda\Big[\bigcup_{i=1}^k \{B_{|\cdot|_\rho}(0, 1+\delta) + x_i\}\Big] \\ &\subset \lambda\Big[\bigcup_{i=1}^k \{(1+\varepsilon)B_\rho + x_i\}\Big] \\ &= \lambda(1+\varepsilon)\Big[\bigcup_{i=1}^k \{B_\rho + \frac{x_i}{1+\varepsilon}\}\Big] \end{aligned}$$

Thus, $\alpha_\rho(D) \le \lambda + \lambda\varepsilon < \lim_k d_\rho^k(D) + \varepsilon + \lambda\varepsilon$ for every $\varepsilon > 0$. Hence $\alpha_\rho(D) \le \lim_k d_\rho^k(D)$ as desired.

Remark. Condition 2) in Theorem 2 is equivalent to Δ_2 condition in Musielak–Orlicz spaces (see [8]). The measure in the definition of Musielak–Orlicz spaces must be $sigma$-finite and atomless.

The following theorem applies to any s-convex, $0 < s \le 1$, modular function, not just to Orlicz spaces.

Theorem 3. *Let ρ be left continuous s-convex modular, $0 < s \le 1$. Then*

$$\lim_k d_\rho^k(D) = \alpha_\rho(D).$$

Proof. We need to show that the assumptions of Theorem 2 are satisfied. By the [7], we already know that $R_\rho(X_\rho) = +\infty$. In the s-convex case, the Conditions 2) and 3) in Theorem 2 are not necessary to prove that D_2 is a bounded set. Since in this case $B_{|\cdot|_\rho}(0, 1) = B_\rho$ (here the left continuity is needed) is a ρ-bounded set ($a_n \to 0$, $\rho(x_n) \leq 1$ then for $n \geq n_0$, $\rho(a_n x_n) \leq a_n^s \rho(x_n) \leq a_n^s \to 0$). Consequently, D_1 is a ρ-bounded set and this implies that D_2 is a ρ-bounded set. To see this, consider $a_n \to 0$, $h_n \in D_2$, then $\lambda g_n + h_n = f_n \in D$, $\rho(g_n) \leq 1$.

Note that, by I.3,

$$\begin{aligned} \rho(a_n h_n) &= \rho(a_n(f_n - \lambda g_n)) \\ &= \rho(\frac{1}{2}[2a_n f_n + (-2\lambda a_n g_n)]) \\ &\leq \rho(2a_n f_n) + \rho(2\lambda a_n g_n). \end{aligned}$$

Since D_2 is ρ-bounded and λB_ρ is ρ-bounded, the last two terms tend to 0 as $n \to \infty$.

To complete the proof, we need to establish Condition 1) in Theorem 2, for any s-convex modular ρ.

First observe that $B_{|\cdot|_\rho}(0, 1+\varepsilon) = (1+\varepsilon)^{1/s} B_\rho$. Let $\varepsilon > 0$ be fixed, choose $\delta > 0$ such that $(1+\delta)^{1/s} \leq 1 + \varepsilon$. Then

$$B_{|\cdot|_\rho}(0, 1+\delta) = (1+\delta)^{1/s} B_\rho \subset (1+\varepsilon) B_\rho.$$

Remark. The above theorem is an improvement of Theorem 2 in [1], which clarifies the solution in the s-convex case without any Δ_2-condition.

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