Minimal Projections with respect to Numerical Radius

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Abstract

In this paper we survey some results on minimality of projections with respect to numerical radius. We note that in the cases L^p, $p = 1, 2, \infty$, there is no difference between the minimality of projections measured either with respect to operator norm or with respect to numerical radius. However, we give an example of a projection from l_3^p onto a two-dimensional subspace which is minimal with respect to norm, but not with respect to numerical radius for $p \neq 1, 2, \infty$. Furthermore, utilizing a theorem of Rudin and motivated by Fourier projections, we give a criterion for minimal projections, measured in numerical radius. Additionally, some results concerning strong unicity of minimal projections with respect to numerical radius are given.

1 Introduction

A projection from a normed linear space X onto a subspace V is a bounded linear operator $P : X \to V$ having the property that $P_{|_V} = I$. P is called a *minimal projection* if $\|P\|$ is the least possible. Finding a minimal projection of the least norm has its obvious connection to approximation theory, since for any $P \in \mathcal{P}(X, V)$, the set of all projections from X onto V, and $x \in X$, from the inequality:

$$\|x - Px\| \leq (\|Id - P\|)\, dist(x, V) \leq (1 + \|P\|)\, dist(x, V), \tag{1}$$

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one can deduce that Px is a good approximation to x if $\|P\|$ is small. Furthermore, any minimal projection P is an extension of Id_V to the space X of the smallest possible norm, which can be interpreted as a Hahn-Banach extensions. In general, a given subspace will not be the range of a projection of norm 1, and the projection of least norm is difficult to discover even if its existence is known *a priori*. For example, the minimal projection of $C[0,1]$ onto the subspace Π_3 of polynomials of degree ≤ 3 is unknown. For an explicit determination of the projection of minimal norm from the subspace $C[-1,1]$ onto Π_2, see [8]. However, it is known that, see [10], for a Banach space X and subspace $V \subset X$, $V = Z^*$ for some Banach space Z, then there exists a minimal projection $P : X \to Z$. A well known example of a minimal projection, [13], is Fourier projection $F_m : C(2\pi) \to \Pi_M := \text{span}\{1, \sin x, \cos x, \ldots, \sin mx, \cos mx\}$ defined as

$$F_m(f) = \sum_{k=0}^{m} \alpha_k \cos kx + \sum_{k=0}^{m} \beta_k \sin kx \tag{2}$$

where α_k, β_k are Fourier coefficients and $C(2\pi)$ denotes $2\pi-$periodic, real-valued functions equipped with the sup norm. For uniqueness of minimality of Fourier projection also see [19]. Let X be a Banach space over $\mathbb{R}$ or $\mathbb{C}$. We write $B_X(r)$ for a closed ball with radius $r > 0$ and center at 0 (B_X if $r = 1$) and S_X for the unit sphere of X. The dual space of X is denoted by X^* and the Banach algebra of all continuous linear operators going from X into a Banach space Y is denoted by $B(X,Y)$ ($B(X)$ if $X = Y$).

The numerical range of a bounded linear operator T on X is a subset of a scalar field, constructed in such a way that it is related to both algebraic and norm structures of the operator, more precisely:

Definition 1.1. *The* n*umerical range* $T \in \vec{B}(X)$ *is defined by*

$$W(T) = \{x^*(Tx) : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}. \tag{3}$$

Notice that the condition $x^*(x) = 1$ gives us that x^* is a norm attaining functional.

The concept of a numerical range comes from Toepliz's original definition of the *field of values* associated with a matrix, which is the image of the unit sphere under the quadratic form induced by the matrix A:

$$F(A) = \{x^* Ax : \|x\| = 1, x \in \mathbb{C}^n\}, \tag{4}$$

where x^* is the original conjugate transform and $\|x\|$ is the usual Euclidean norm. It is known that the classical numerical range of a matrix always contains the spectrum, and as a result study of numerical range can help understand properties that depend on the location of the eigenvalues such as stability and non-singularity of matrices. In case A is a normal matrix, then the numerical range is the polygon in the complex plane whose vertices are eigenvalues of A. In particular, if A is hermitian, then the polygon reduces to the segment on the real axis bounded by the smallest and largest eigenvalue, which perhaps explains the name numerical range.

The *numerical radius* of T is given by

$$\|T\|_w = \sup\{|\lambda| : \lambda \in W(T)\}. \tag{5}$$

Clearly $\|T||_w$ is a semi-norm on $B(X)$ and $\|T\|_w \leq \|T||$ for all $T \in B(X)$. For example, if we consider $T : \mathbb{C}^n \to \mathbb{C}^n$ as a right shift operator

$$T(f_1, f_2 \ldots, f_n) = (0, f_1, f_2, \ldots, f_{n-1})$$

then $\langle Tf, f\rangle = f_1\overline{f}_2 + f_2\overline{f}_3 + \ldots f_{n-1}\overline{f}_n$ and consequently to find $\|T\|_w$ we must find $\sup\{|f_1||f_2|+\cdots+|f_{n-1}||f_n|\}$ subject to the condition $\sum_{i=1}^{n} |f_i|^2 = 1$. The solution to this "Lagrange multiplier" problem is

$$\|T\|_w = \cos\left(\frac{\pi}{n+1}\right). \tag{6}$$

The *numerical index* of X is then given by

$$n(X) = \inf\left\{\|T\|_w : T \in S_{B(X)}\right\}. \tag{7}$$

Equivalently, the numerical index $n(X)$ is the greatest constant $k \geq 0$ such that $k\|T\| \leq \|T\|_w$ for every $T \in B(X)$. Note also that $0 \leq n(X) \leq 1$ and $n(X) > 0$ if and only if $\|\cdot\|_w$ and $\|\cdot\|$ are equivalent norms. The concept of numerical index was first introduced by Lumer [14] in 1968. Since then much attention has been paid to the constant of equivalence between the numerical radius and the usual norm of the Banach algebra of all bounded linear operators of a Banach space. Two classical books devoted to these concepts are [7] and [6]. For more recent results we refer the reader to [4],[15],[16] and [11].

In this paper, we study minimality of projections with respect to numerical radius. Since operator norm of T is defined as $\|T\| = \sup|\langle Tx, y\rangle|$ with $(x, y) \in B(X) \times B(X^*)$, while numerical radius $\|T\|_w = \sup|\langle Tx,\ y\rangle|$ with $(x, y) \in B(X) \times B(X^*)$ and $\langle x, y\rangle = 1$, $\|T\|$ is bilinear and $\|T\|_w$ is quadratic in nature. However, $\|T\|_w \leq \|T\|$ implies that there are more spaces for which $\|T\| \geq 1$ but $\|T\|_w = 1$.

Furthermore, if T is a bounded linear operator on a Hilbert space H, then the numerical radius takes the form

$$\|T\|_w = \sup\{|\langle Tx, x\rangle| : \|x\| = 1\}. \tag{8}$$

This follows from the fact that for each linear functional x^* there is a unique $x_0 \in H$ such that $x^*(x) = \langle x, x_0\rangle$ for all $x \in H$. Moreover, if T is self-adjoint or a normal operator on a Hilbert space H, then

$$\|T\|_w = \|T\|. \tag{9}$$

Also, if a non-zero $T : H \to H$ is self-adjoint and compact, then T has an eigenvalue λ such that

$$\|T\|_w = \|T\| = \lambda. \tag{10}$$

These properties of numerical radius together with the desirable properties of diagonal projections from Hilbert spaces onto closed subspaces proved motivation to investigate minimal projections with respect to numerical radius.

2 Characterization of Minimal Numerical Radius Projections

In [1] a characterization of minimal numerical radius extension of operators from a Banach space X onto its finite dimensional subspace $V = [v_1, v_2, \dots, v_n]$ is given. To express this theorem, we first set up our notation.

Notation 2.1. *Let* $T = \sum_{i=1}^{n} u_i \otimes v_i : V \to V$ *where* $u_i \in V^*$ *and its extension to* X *is denoted by* $\widetilde{T} : X \to V$ *and defined as*

$$\widetilde{T} = \sum_{i=1}^{n} \tilde{u}_i \otimes v_i, \tag{11}$$

where $\widetilde{u}_i \in X^*$.

Definition 2.2. *Let X be a Banach space. If $x \in X$ and $x^* \in X^*$ are such that*

$$|\langle x, x^* \rangle| = \|x\| \|x^*\| \neq 0, \tag{12}$$

then x^* is called an extremal *of x and written as $x^* = ext\, x$. Similarity, x is an extremal of x^*. We call $(ext\, y, y) \in S_{X^{**}} \times S_{X^*}$ a* diagonal extremal pair *for $\widetilde{T} \in B(X, V)$ if*

$$\langle \widetilde{T}^{**} x, y \rangle = \|\widetilde{T}\|_w, \tag{13}$$

where $\widetilde{T}^{**} : X^{**} \to V$ is the second adjoint extension of $\widetilde{T}$ are $V = [v_1, \ldots, v_n] \subset X$. In other words, the map $\widetilde{T}$ has the expression $\widetilde{T} = \sum_{i=1}^{n} \tilde{u}_i \otimes v_i : X \to V$ and

$$\widetilde{T} x = \sum_{n=1}^{n} \langle x, \tilde{u}_i \rangle v_i \tag{14}$$

where $\tilde{u}_i \in X^*$, $v_i \in V$ and $\langle x, \tilde{u}_i \rangle$ denotes the functional $\tilde{u}_i$ is acting on x and

$$\widetilde{T}^{**} x = \sum_{i=1}^{n} \langle u_i, x \rangle v_i, \tag{15}$$

$u_i \in X^{***}$, $v_i \in V$, $x \in X^{**}$.

The set of all diagonal extremal pairs will be denoted by $\mathcal{E}_w(\widetilde{T})$ and defined as:

$$\mathcal{E}_w(\widetilde{T}) = \left\{ (ext\, y, y) \in S_{X^{**}} \times S_{X^*} : \ \|\widetilde{T}\|_w = \sum_{i=1}^{n} \langle ext\, y, u_i \rangle \cdot \langle v_i, y \rangle \right\}. \tag{16}$$

Note that to each $(x, y) \in X^{**} \times X^*$ we associate the rank-one operator $y \otimes x : X \to X^{**}$ given by

$$(y \otimes x)(z) = \langle z, y \rangle x \qquad \text{for } z \in X. \tag{17}$$

Accordingly, to each $(x, y) \in \mathcal{E}_w(\widetilde{T})$ we can associate the rank-one operator $y \otimes ext\, y : X \to X^{**}$ given by

$$(y \otimes ext\, y)(z) = \langle z, y \rangle ext\, y. \tag{18}$$

By $\mathcal{E}(\widetilde{T})$ we denote the usual set of all extremal pairs for $\widetilde{T}$ and

$$\mathcal{E}(\widetilde{T}) = \left\{ (x, y) \in S_{X^{**}} \times S_{X^*} : \|\widetilde{T}\| = \sum_{i=1}^{n} \langle x, u_i \rangle \cdot \langle v_i, y \rangle \right\}. \tag{19}$$

In case of diagonal extremal pairs we require $|\langle ext\, y, y\rangle| = 1$.

Definition 2.3. *Let* $T = \sum_{i=1}^{n} u_i \otimes v_i : V \to V = [v_1, v_2, \ldots, v_n] \subset X$, *where* $u_i \in V^*$. *Let* $\widetilde{T} : \sum i = 1^n \tilde{u}_i \otimes v_i : X \to V$ *be an extension of* T *to all of* X. *We say* $\widetilde{T}$ *is a* minimal numerical extension *of* T *if*

$$\|\widetilde{T}\| = \inf \left\{ \|S\|_w : S : X \to V \ ; \ S_{|V} = T \right\}. \tag{20}$$

Clearly $\|T\|_w \le \|\widetilde{T}\|_w$.

Theorem 2.4. *([1])* $\widetilde{T}$ *is a minimal radius-extension of* T *if an only if the closed convex hull of* $\{y \otimes x\}$ *where* $(x, y) \in \mathcal{E}_w(\widetilde{T})$ *contains an operator for which* V *is an invariant subspace.*

Theorem 2.5. P *is a minimal projection from* X *onto* V *if and only if the closed convex hull of* $\{y \otimes x\}$, *where* $(x, y) \in \mathcal{E}_w(P)$ *contains an operator for which* V *is an invariant subspace.*

Proof. By taking $T = I$ and $\widetilde{T} = P$ one can appropriately modify the proof given in [1] without much difficulty. The problem is equivalent to the best approximation in the numerical radius of a fixed operator from the space of operator

$$\mathcal{D} = \{\Delta \in \mathcal{B} : \Delta = 0 \text{ on } V\} = sp\{\delta \otimes v : \delta \in V^{\perp}; v \in V\}.$$

One of the main ingredients of the proof is Singer's identification theorem ([20], Theorem 1.1 (p.18) and Theorem 1.3 (p.29)) of finding the minimal operator as the error of best approximation in C(K) for K Compact. In the case of numerical radius, one considers $K_w = K \cap Diag = \{(x, y) \in B(X^{**}) \times B(X^*) : x = ext(y) \text{ or } x = 0\}$ and shows K_w is compact. Thus the set $\mathcal{E}(P)$, being the set of points where a continuous (bilinear) function achieves its maximum on a compact set, is not empty. For further details see [1]. □

Theorem 2.6. *(When minimal projections coincide) In case* $X = L^p$ *for* $p = 1, 2, \infty$, *the minimal numerical radius projections and the minimal operator norm projections coincide with the same norms.*

Proof. In case of L^2, for any self-adjoint operator, we have

$$\|P\| = \|P\|_w = |\lambda|, \tag{21}$$

where λ is the maximum (in modulus) eigenvalue. In this case,

$$\|P\| = \|P\|_w = |\langle Px, x\rangle|, \tag{22}$$

where x is a norm-1 "maximum" eigenvector.

When $p = 1, \infty$, it is well known that $n(L^p) = 1$ ([7], section 9) thus

$$\|P\| = \|P\|_w. \tag{23}$$

□

Example 2.7. *The projection $P : l_3^p \to [v_1, v_2] = V$ where $v_1 = (1,1,1)$ and $v_2 = (-1,0,1)$ is minimal with respect to the operator norm, but not minimal with respect to numerical radius for $1 < p < \infty$ and $p \neq 2$. Let us denote by P_o, P_m projections minimal with respect to operator norm and numerical radius respectively. In other words*

$$\begin{aligned} \|P_o\| &= \inf\{\|P\| : P \in \mathcal{P}(X,V)\} \\ \|P_m\|_w &= \inf\{\|P\|_w : P \in \mathcal{P}(X,V)\}. \end{aligned}$$

Note that

$$P_o(f) = u_1(f)v_1 + u_2(f)v_2 \quad \textit{and} \quad P_m(f) = z_1(f)v_1 + z_2(f)v_2. \tag{24}$$

Then it is easy to see that

$$\begin{aligned} u_1 = z_1 &= \left(-\frac{1}{2}, 0, \frac{1}{2}\right) \\ u_2 &= \left(\frac{1-d}{2}, d, \frac{1-d}{2}\right) \\ z_2 &= \left(\frac{1-g}{2}, g, \frac{1-g}{2}\right), \end{aligned}$$

and for $p = \frac{4}{3}$ it is possible to determine g and d to conclude $\|P_o\| = 1.05251$ while $\|P_m\|_w = 1.02751$, thus $\|P_o\| \neq \|P_m\|_w$.

V. P. Odinec in [18] (see also [17], [12]) proves that minimal projections of norm greater than one from a three-dimensional Banach space onto any of its two-dimensional subspaces are unique. Thus in the above example, the projection from l_3^p onto a two-dimensional subspace not only proves the fact that $\|P_o\| \neq \|P_m\|_w$ for $p \neq 1, 2, \infty$, here once again we have the uniqueness of the projections.

3 Rudin's Projection and Numerical Radius

One of the key thoerems on minimal projections is due to W. Rudin ([21] and [22]) The setting for his theorem is as follows. X is a Banach space and G is a compact topological group. Defined on X is a set $\mathcal{A}$ of all bounded linear bijective operators in a way that $\mathcal{A}$ is algebraically isomorphic to G. The image of $g \in G$ under this isomorphism will be denoted by T_g. We will assume that the map $G \times X \to X$ defined as $(g, x) \mapsto T_g x$ is continuous. A subspace V of X is called *G-invariant* if $T_g(V) \subset V$ for all $g \in G$ and a mapping $S : X \to X$ is said to *commute* with G if $S \circ T_g = T_g \circ S$ for all $g \in G$. In case $\|T_g\| = 1$ for all $g \in G$, we say g acts on G by *isometries*.

Theorem 3.1. *([22]) Let G be a compact topological group acting by isomorphism on a Banach space X and let V be a complemented G–invariant subspace of X. If there exists a bound projection P of X onto V, then there exists a bounded linear projection Q of X onto V which commutes with G.*

The idea behind the proof of the above theorem is to obtain Q by averaging the operators $T_{g^{-1}}PT_g$ with respect to Haar measure μ on G. i.e.,

$$Q(x) := \int_G (T_{g^{-1}}PT_g)(x)\, d\mu(g). \tag{25}$$

Now assume X has a norm which contains the maps $\mathcal{A}$ to be *isometries* and all of the hypotheses in Rudin's theorem are satisfied, then one can claim the following stronger version of Rudin's theorem :

Corollary 3.2. *If there is a* unique *projection $Q : X \to V$ which commutes with G, then for any $P \in \mathcal{P}(X, V)$, the projection*

$$Q(x) = \int_G (T_{g^{-1}}PT_g)(x)\, d\mu(g), \tag{26}$$

is a minimal projection *of X onto V.*

Theorem 3.3. *([3]) Let $\mathcal{A}$ be a set of all bounded linear bijective operators on X such that $\mathcal{A}$ is algebraically isomorphic to G. Suppose that all of the hypotheses of Rudin's theorem above are satisfied and the maps in $\mathcal{A}$ are isometries. If P is any projection in the numerical radius of X onto V, then the projection Q defined as*

$$Q(x) = \int_G \left(T_{g^{-1}} P T_g\right)(x)\, d\mu(g) \tag{27}$$

satisfies $\|Q\|_w \leq \|P\|_w$.

Proof. Consider $\|Q\|_w = \sup\{|x^*(Qx)| : x^*(Qx) \in W(Q)\}$, where $W(Q)$ is the numerical range of Q. Notice that

$$\begin{aligned} |x^*(Qx)| &= \left| x^* \int_G \left(T_{g^{-1}} P T_g\right)(x)\, d\mu(g) \right| \\ &\leq \int_G \left| \left(x^* \circ T_{g^{-1}}\right) P(T_g x) \right|\, d\mu(g). \end{aligned} \tag{28}$$

But $\|x\| = 1$ and $\|x^*\| = 1$ which implies that $\|T_g x\| = 1$ and $\|x^* T_{g^{-1}}\| = 1$, moreover,

$$1 = x^*(x) = x^* T_{g^{-1}}(T_g x) \implies |x^*(Qx)| \leq \|P\|_w. \tag{29}$$

Consequently, $\|Q\|_w \leq \|P\|_w$ which proves Q is a minimal projection in numerical radius. □

Theorem 3.4. *([3]) Suppose all hypotheses of the above theorem are satisfied and that there is exactly one projection Q which commutes with G. Then Q is a minimal projection with respect to numerical radius.*

Proof. Let $P \in \mathcal{P}(X, V)$. By the properties of Haar measure, Q_p given in the above theorem commutes with G. Since there is exactly one projection which commutes with G, $Q_p = Q$ and $\|Q\|_w \leq \|P\|_w$ as desired. □

Remark 3.5. *In [3] it is shown that if G is a compact topological group acting by isometries on a Banach space X and if we let*

$$\psi : B(X) \to [0, +\infty], \tag{30}$$

be a convex function which is lower semi-continuous in the strong operator topology and if one further assumes that

$$\psi\left(g^{-1} \circ P \circ g\right) \leq \psi(P), \tag{31}$$

for some $P \in B(X)$ and $g \in G$, then $\psi(Q_P) \leq \psi(P)$. This result leads to calculation of minimal projections not only with respect to numerical radius but also with respect to p-summing, p-nuclear and p-integral norms. For details see [3].

4 An Application

Let $C(2\pi)$ denote the set of all continuous 2π-periodic functions and Π_n be the space of all trigonometric polynomials of order $\leq n$ (for $n \geq 1$).

The *Fourier projection* $F_n : C(2\pi) \to \Pi_n$ is defined by

$$F_n(f) = \sum_{k=0}^{2n} \left(\int_0^{2\pi} f(t) g_n(t) dt \right) g_k, \tag{32}$$

where $(g_k)_{k=0}^{2n}$ is an orthonormal basis in Π_n with respect to the scalar product

$$\langle f, g \rangle = \int_0^{2\pi} f(t) g(t) dt. \tag{33}$$

Lozinskii in [13] showed that F_n is a minimal projection in $\mathcal{P}(C(2\pi), \Pi_n)$. His proof is based on the equality which states that for any $f \in C(2\pi)$, $t \in [0, 2\pi]$ and $P \in \mathcal{P}(C(2\pi), \Pi_n)$, we have

$$F_n f(t) = \frac{1}{2\pi} \int_0^{2\pi} \left(T_{g^{-1}} P T_g f \right)(t) \, d\mu(g). \tag{34}$$

Here μ is a Lebesgue measure and $(T_g f)(t) = f(t+g)$ for any $g \in \mathbb{R}$. This equality is called Marcinkiewicz equality ([9] p.233).

Notice that F_n is the only projection that commutes with G, where $G = [0, 2\pi]$ with addition mod 2π. In particular, F_n is a minimal projection with respect to numerical radius.

Since we know the upper and lower bounds on the operator norm of F_n, more precisely ([9] p.212)

$$\frac{4}{\pi^2} ln(n) \leq \|F_n\| \leq ln(n) + 3. \tag{35}$$

From the theorem (when minimal projections coincide) we know that in cases of L^p, $p = 1, \infty$, the numerical radius projections and the operator norm projections are equal. Since $C(2\pi) \in L^\infty$, we also have lower and upper bounds for the numerical radius of Fourier projections, i.e.,

$$\frac{4}{\pi^2} ln(n) \leq \|F_n\|_w \leq ln(n) + 3. \tag{36}$$

Remark 4.1. *Lozinskii's proof of the minimality of F_n is based on Marcinkiewicz equality. However, the Marcinkiewicz equality holds true if one replaces $C(2\pi)$ by $L^p[0, 2\pi]$ for $1 \leq p \leq \infty$ or Orlicz space $L^\phi[0, 2\pi]$ equipped with Luxemburg or Orlicz norm provided ϕ satisfies the suitable Δ_2 condition. Hence, Theorem 3.3 can be applied equally well to numerical radius or norm in Banach operator ideals of p-summing, p-integral,p-nuclear operators generated by L^p-norm or the Luxemburg or Orlicz norm. For further examples see [3].*

5 Strongly Unique Minimal Extensions

In [18] (see also [17]) it is shown that a minimal projection of the operator norm greater than one from a three dimensional real Banach space onto any of its two dimensional subspace is the unique minimal projection with respect to the operator norm. Later in [12] this result is generalized as follows:
Let X is a three dimensional real Banach space and V be its two dimensional subspace. Suppose $A \in B(V)$ is a fixed operator. Set

$$\mathcal{P}_A(X, V) = \{P \in B(X, V) : \ P\mid_V = A\}$$

and assume $\| P_0 \| > \| A \|$, if $P_o \in \mathcal{P}_A(X, V)$ is an extension of minimal operator norm. Then P_o is *a strongly unique minimal extension with respect to operator norm.*
In other words there exists $r > 0$ such that for all $P \in \mathcal{P}_A(X, V)$ one has

$$\|P\| \geq \|P_o\| + r\,\|P - P_o\|.$$

Definition 5.1. *We say an operator $A_o \in \mathcal{P}_A(X, V)$ is a strongly unique minimal extension with respect to numerical radius if there exists $r > 0$ such that*

$$\|B\|_w \geq \|A_o\|_w + r\,\|B - A_o\|_w$$

for any $B \in \mathcal{P}_A(X, V)$.

A natural extension of the above mentioned results to the case of numerical radius $\| \cdot \|_w$ was given in [2].

Theorem 5.2. *([2]) Assume that X is a three dimensional real Banach space and let V be its two dimensional subspace. Fix $A \in B(V)$ with $\|A\|_w > 0$. Let*

$$\lambda_w^A = \lambda_w^A(V, X) = \inf\{\|B\|_w : B \in \mathcal{P}_A(X, V)\} > \|A\|,$$

where $\|A\|$ denotes the operator norm. Then there exist exactly one $A_o \in \mathcal{P}_A(X, V)$ such that

$$\lambda_w^A = \|A_o\|_w.$$

Moreover, A_o is the strongly minimal extension with respect to numerical radius.

Notice that if we take $A = id_V$ then $\|A\|_w = \|A\| = 1$. In this case Theorem (5.2) reduces to the following theorem:

Theorem 5.3. *([2]) Assume that X is a three dimensional real Banach space and let V be its two dimensional subspace. Assume that*

$$\lambda_w^{id_V} > 1.$$

Then there exist exactly one $P_o \in \mathcal{P}(X, V)$ of minimal numerical radius. Moreover , P_o is a strongly unique minimal projection with respect to numerical radius. In particular P_o is the only one minimal projection with respect to the numerical radius.

Remark 5.4. *([2]) Notice that in Theorem (5.2) the assumption that $\|A\| < \lambda_w^A$ is essential. Indeed, let $X = l_\infty^{(3)}$, $V = \{x \in X : x_1 + x_2 = 0\}$ and $A = id_V$. Define*

$$P_1 x = x - (x_1 + x_2)(1, 0, 0)$$

and

$$P_2 x = x - (x_1 + x_2)(0, 1, 0).$$

It is clear that

$$\|P_1\| = \|P_1\|_w = \|P_2\| = \|P_2\|_w = 1$$

and $P_1 \neq P_2$. Hence, there is no strongly unique minimal projection with respect to numerical radius in this case.

Remark 5.5. *([2]) Theorem (5.3) cannot be generalized for real spaces X of dimension $n \geq 4$. Indeed let $X = l_\infty^{(n)}$, and let $V = ker(f)$, where $f = (0, f_2, ..., f_n) \in l_1^{(n)}$ satisfies $f_i > 0$ for $i = 2, ..., n$, $\sum_{i=2}^n f_i = 1$ and $f_i < 1/2$ for $i = 1, ..., n$. It is known (see e.g. [5], [17]) that in this case*

$$\lambda(V, X) = 1 + (\sum_{i=2}^{n} f_i/(1 - 2f_i))^{-1} > 1,$$

where

$$\lambda(V, X) = \inf\{\|P\| : P \in (X, V)\}.$$

By [1], $\lambda(V, X) = \lambda_w^{id_V}$. Define for $i = 1, ..., n$ $y_i = (\lambda(V, X) - 1)(1 - 2f_i)$. Let $y = (y_1, ..., y_n)$ and $z = (0, y_2, ..., y_n)$. Consider mappings P_1, P_2 defined by

$$P_1 x = x - f(x)y$$

and

$$P_2 x = x - f(x)z$$

for $x \in l_\infty^{(n)}$. It is easy to see that $P_i \in \mathcal{P}(X, V)$, for $i = 1, 2$, $P_1 \neq P_2$. By ([17] p. 104) $\|P_i\| = \|P_i\|_w = \lambda(V, X) = \lambda_w^{id_V}$. for $i = 1, 2$.

Remark 5.6. *Theorem(5.3) is not valid for complex three dimensional spaces. For details see [2].*

For Kolmogorov type criteria concerning approximation with respect to numerical radius, we refer the reader to [2].

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